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This project endeavored to develop a high school course for use in the classroom and for use with a computer-controlled system of programed lessons. Many difficulties such as those in the display and control of figures were encountered in trying to organize and prepare the material for the computer. The conclusion reached was that the solution to the problem is possible, but a reasonably extensive series of lessons will require a larger computer than those available. Furthermore, the heavy task ot rogram preparation requires a larger staff than that supported by th project. This report also proposes an outline for a one semester high school course in the area of geometric transformations. The short term of the project did not permit arrangements for teaching the materials to larger staff than that supported by the project. This report also proposes an outline for a one high school students. (RP)

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Project No. H257

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Developing New Materials for High School Geometry

Dana S. Scott

Stanford University
Stanford, California

June 1, 1968

The research reported herein was performed pursuant to a contract with the Office of Education, U.S. Department of Health, Education, and Welfare. Contractors undertaking such projects under Government sponsorship are encouraged to express freely their professional judgment in the conduct of the project. Points of view or opinion stated do not, therefore, necessarily represent official Office of Education position or policy.

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We would also like to thank David D. Struthers of the Palo Alto Unified School District for discussing the possibility of teaching the experimental course in the framework of their high school program. We are extremely sorry that pressures of our other academic commitments prevented us from extending the project another year to be able to carry out this plan.

PART I. INTRODUCTION

§1. Summary

Geometric transformations have for some time been regarded as an area suitable for enrichment and development of geometry both for high school and teacher training curricula. This project studied the material from the point of view of the high school level. The plan was to develop a high school course both for use in the classroom and for use with a computer controlled system of programmed lessons.

Many difficulties were encountered in trying to organize and prepare the material for the machine. One of the main problems had to do with the display and control of figures. The conclusion reached was that the solution to the problem is possible, but a reasonably extensive series of lessons would require a larger computer than we had available. Furthermore, the heavy task of program preparation required a larger staff than we could support on this project. Certainly the ever increasing number of students will force us soon to develop these computer-assisted systems. A certain amount of technical development in graphical display still remains to be done for the kind of problem we considered, but mainly it requires a very concerted programming effort that would have to be done on a fairly large scale by a full-time staff.

In the meantime, work needs to be done in developing the proper materials. For the area of geometric transformations we propose in this report an outline for a one semester high school course that remains to be tested. Considering the difficulties with the work on the machine, it is clear now that our project should have concentrated solely on the course material. The short term of the project did not allow us to make arrangements for teaching the material to high-school students, something that surely should be done in the near future. This conclusion should not, however, be taken as a negative result concerning feasibility of computer-assisted lessons. Such lessons are feasible, but should only be attempted after adequate testing of classroom material.

§2. Background and Description of Report

The purpose of the project was to study the possibility of developing materials on geometric transformations suitable for the high school level. In connection with consideration of the subject matter itself another aim was to experiment with the feasibility of programming this material for use with a computer-assisted course of lessons. The reasons why geometric transformations present themselves as an interesting area for such a study are many. In the first place they enrich the usual geometry course with easily grasped ideas leading to other important areas such as vectors and groups. In the second place they provide a powerful tool for problem solving which demonstrates how an abstract mathematical idea can be put to work, a third reason is that the visual character of the notion lends itself to computer display: possibly the computer even has a special advantage in that the figures can be accurately modified ("transformed") before the subject's eyes. On paper or on the blackboard such transformations can only be imagined or illustrated for only a limited number of figures. Thus the machine could become a highly flexible means of experimentation.

In this report in Part II there are three sections. The first by Michael H. Millar gives a detailed outline of a semester course that we feel is suitable for the high school curriculum. In the second section by Peter Belew a description is given of how we actually programmed two of the lessons for the machine. In connection with this, the reader will find in the appendices the explicit texts used and examples of the machine displayed figures. The third section by

John Lennie presents views on the organizational problems of such computer programs dealing with interaction between the subject and the machine. The conclusions reached are presented in Part III of the report.

§3. Methods

Not included in this report because of the obvious lack of documentation is a description of the many months of discussion and experimentation. The discussion concerned the nature of the material on transformations and the attempt to isolate the essential ideas. The plan of trying to program the lessons for the machine was excellent for this purpose because of the difficulty of getting the material into a machine format. One really must question what is important because he cannot rely on the inspiration of the classroom to find out how to convey the ideas. Also we were faced with such questions as: "What is a geometric figure?" The answer still is not clear. Certainly a figure is more than just the set of its points. The human subject always conceptualizes and isolates parts and features of a figure to suit his needs. How to represent such organization of data in the machine in a flexible way is an extremely stimulating problem. It was consideration of this problem that lead to considerable experimentation on the machine. We tried out several methods of representing and controlling figures and found, unfortunately, that size of the particular installation precluded a dynamic use of figures in connection with the lessons. We did, however, develop methods of constructing figures which were incorporated into the lessons actually programmed.

For the reader to get an idea of what we had to work with at Stanford, we present next a description of the Computer-based Instruction Laboratory which is directed by Professor Patrick Suppes.

The key component in the Laboratory is a PDP-1 computer with 32 k* 18-bit words of 5.35 μsec core memory, a fast drum and a large disk. It operates under a time-sharing system which allows many users to carry on independent dialogues with it, though to each user it seems that his computer has only 12 k words of core, organized in 4 k units, and there is no drum. The reason for this is that all users share the same three units of core and when one user refers to a unit belonging to another there is a delay while one unit is saved on the drum, then replaced from the drum by the unit referenced. Programs take their turn in a cycling queue and if a user has not finished his computing in 64 m sec the queue is advanced to give the next user his turn. The time for a complete cycle depends both on the number of users and the number who are active. The word 'dialogue' was used advisedly because in such an environment a user is likely to spend most of his time between short spells of computing (perhaps one or two time-units) in wondering what to do next. The system passes over him in the queue until he is ready. For a typical load with perhaps eight users the cycle might be about one second, exceptionally going up to three seconds.

The system can accommodate up to 20 independent programs each of which is associated with one of 12 Philco Read consoles. Each of these consoles is provided with a 10 × 10 inch TV screen (often referred to as a 'scope'), a light-pen which can sense an illuminated object on the screen and a keyboard with 115 characters, including two cases, some Greek characters and a complete set of logical symbols. (There is a photograph of one of these consoles on page 140. In addition six consoles have a microphone and a speaker which can record and play back spoken messages under computer control.

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*1 k = 2^{10} = 1024

Each program may set up its own time-sharing system using other Philco consoles as available and teletypes. Since applications in geometry clearly require displays, it seems appropriate to say a little more about how they are organized. All the Philco displays are driven by one controller which processes words in core sequentially, interpreting them as commands to position and move the individual electron guns, or switch between vector or character mode. There are absolute positioning and relative positioning commands and 1024 × 1024 discrete positions on each screen. Seven character sizes and three levels of illumination are available. Curves must be approximated by line segments. Because the display makes heavy demands on the system, both to maintain its data and because it absorbs up to 1/4 of the word-cycles, the size of each user's display is limited. Although the Philco displays are quite good there are at least two ways they could be improved, (1) by including conic generating facilities as has been done at Lincoln Laboratories and (2) by including more commands for the display controller, in particular jumps and subroutines. The light-pen facilities also leave something to be desired because they only supply the coordinates of the end-points of a vector when there is an interrupt.

Several languages are available in the system. At the basic level there is a good assembly language, PASS, which was used for the Independent Student Program. The most convenient higher-level language with excellent facilities for manipulating the display (albeit in a non-dynamic way) and strings is TSA and this was used for the

Multiple-choice Program. TSA is an Algol-like language defined by syntax equations to a compiler-compiler and programs are translated to a stack language which is executed interpretively. Thus it is both slow and space-limited to small programs with relatively little data. In addition it has no facilities for dynamically altering displayed data. It would have greatly eased the programming of the Independent Student Program if it had been feasible to use TSA. However, running such a program even in assembly language would make heavier demands on the system than any time-shared programs do at present.

PART II. FINDINGS AND ANALYSIS

§1. A High-School Course on Transformations

by

Michael H. Millar

At a recent joint meeting of the Mathematical Association of America and the National Council of Teachers of Mathematics (at Houston, Texas on Saturday, January 28, 1967) a dozen or so speakers stressed in different ways the interest of transformations for high school geometry. The consensus seemed to be that high school geometry as a subject was worth saving. Further, it was agreed that geometric transformations not only made the course more interesting and meaningful and gave the student a powerful problem-solving instrument, but they also helped relate the geometric ideas with other mathematical concepts (e.g., function, vector space, groups.) It was also made clear how desirable it would be to gradually introduce these ideas throughout the whole mathematical curriculum from grades K to 12. At the present time such a proposal may be a bit optimistic, though various new programs are, in fact, trying to incorporate many of these concepts. Incorporation into the college program is, of course, relatively easy and seems to be essential for the geometric education of prospective teachers. What appears to us to be currently the main problem is the presentation of the material at a proper secondary-school level in a way that the present teachers and students can grasp the interest and usefulness of the ideas in relation to the geometry that is now being taught.

Discussion of the material on transformations

One way in which material on geometric transformations could be presented to a high school audience would be to organize the concept of a rigid transformation in the Euclidean plane around the notion of an axial reflection in the plane. An outline of some possible topics for a one-semester course based on rigid transformations (built up as compositions of axial reflections) and similarity transformations of the plane into itself is given below. This material would be accessible to a student who has completed three years of high school mathematics, including one year of geometry.

1. General Introduction to Transformations: Definition of a transformation: domain and range; one-to-one-ness and ontoness; image of a set; transformations and the set-theoretic operations; compositions and inverses of transformations; conditions for the existence of transformations.

2. Axial Reflections in the Plane: Definition; distance-preserving character; preservation of lines, circles, polygons, conic sections and other geometric figures; invariant points and lines; applications to construction problems and proving theorems (e.g., tangents to a circle from a point outside a circle are equal); applications to the conic sections (e.g., deriving the reflection properties of the ellipse and the parabola).

3. Inverses and Compositions of Axial Reflections: Inverses of axial reflections; compositions of two axial reflections and introduction of translations, central reflections, and rotations; properties of translations, central reflections, and rotations; inverses and compositions of translations, central reflections, and rotations; applications of translations, central reflections, and rotations to construction problems, proving theorems, and deriving properties of the conic sections.

4. General Notion of a Rigid Transformation: Definition of arbitrary rigid transformations; properties of a rigid transformation; existence and uniqueness of rigid transformations (and comparisons to existence and uniqueness features of axial reflections, translations, central reflections, and rotations); inverses and compositions of rigid transformations; rigid transformations and the general notions of congruence and superposition (using the concept of a 'rigid transformation' as a natural generalization of the ordinary Euclidean idea of 'congruence').

5. Rigid Transformations and the Fundamental Decomposition Theorem: Proof that every rigid transformation can be expressed as a composition of not more than three axial reflections; further work with compositions (e.g., the definition of a glide, proof that there are four basic types of rigid transformations, etc.)

6. Rigid Transformations and Alternative Classification Schemes: The concept of orientation; orientation-preserving and orientation-reversing rigid transformations; fixed points; orientation, fixed points, and translations, rotations, axial reflections, and glides.

7. Rigid Transformations and the Theory of Paper-Folding: Initial problems in paper-folding; interpretation of these problems using axial reflections; some 'axioms' for paper-folding; interpretation of these 'axioms' using rigid transformations.

8. Rigid Transformations, Symmetry, and Groups: Definitions of symmetry and symmetry group of a geometric figure; some familiar geometric figures and their symmetry groups; further work with symmetry and various groups of rigid transformations.

9. Similarity Transformations: Definition of a similarity transformation; properties of a similarity transformation; existence and uniqueness of similarity transformations; inverses and compositions of similarity transformations; relations to rigid transformations; applications to construction problems and proving theorems (e.g., inscribing a square or an equilateral triangle in a given triangle); similarity transformations and the general notion of similarity. Applications of similarity transformations to vectors, complex numbers, etc.

The topics described above provide a way in which the rigid and similarity transformations of the Euclidean plane, both in their theoretical and applied contexts, can be studied in a systematic and coherent fashion. Specifically, this material would offer

1) a new method by which a number of familiar and not so familiar construction problems and geometric theorems can be handled. This method is very broad in its scope, and is not restricted to the familiar lines, circles, and polygonal figures of ordinary

high school geometry. For example, many of the properties of the conic sections--which ordinarily require techniques of the calculus--can be handled very neatly using rigid transformations;

2) the necessary concepts through which the familiar geometric notions of congruence, similarity, and symmetry can be made precise and placed in a very general setting;
3) a new way to look at functions. In this work, transformations (or functions) are treated as mathematical entities in themselves, i.e., as things that have properties and can be combined with each other. Here the emphasis moves away from exclusive attention to functional values and from viewing functions as sets of ordered pairs over to a view in which functions are regarded as the mathematical counterparts of various actions performed on certain sets--in this case sets in the Euclidean plane;
4) a powerful tool through the use of which the student can gain a better understanding of such concepts as groups, vectors, complex numbers and others that he will meet in collegiate mathematics.

It is hoped that experience with such a course would give some indication of the problems involved in using transformations to broaden and deepen the student's and teacher's interest and understanding of geometry, and of ways these problems can be solved. The primary emphasis in the work with the high school students would be to explore the possibility of developing a new course in the traditional high school mathematics curriculum that would systematically organize the material on geometric transformations now more or less scattered in the literature. The work would build upon the student's previous geometric experience, and would not represent an attempt to teach geometry at the outset from a transformational viewpoint. However, the question of axiomatizing geometry through the use of rigid and similarity transformations is a very interesting one, and has yet to be worked out in any practical pedagogical way. It is hoped that experience with a high school course in transformations would--if not solve the problem of axiomatization--at least point to some relevant problems that would have to be taken into account in such a reformulation of school geometry.

Implementation of the Semester Course on Geometric Transformations

The material listed under topics (1)-(9) in the previous section would probably be too much for the high school teacher to consider in a one-semester course meeting three or four times weekly. For a basic core, the concepts and theorems listed under (1) through (5) would be essential. This material would introduce the student to the essential ideas concerning transformations in the plane, and would show him the power of the transformation concept in helping solve familiar problems in geometry in more efficient ways, as well as in helping solve problems that cannot be handled at all within the context of ordinary high school geometry. Material listed in topics (6)-(9) represents a number of more or less independent directions a teacher could take once the student has become familiar with the basic core. For example, a teacher interested in exploiting transformations to solve construction problems in geometry might elect to study the notion of a similarity transformation and relevant topics listed under (9). Another teacher, interested in extending the group concept, could treat the material listed under (8).

The day-by-day activities of the student and teacher would be much as they are in the ordinary high school mathematics classroom. There would be definitions to set out and theorems to prove. The study of geometric transformations has at its command a rich source of exercises, and much of the time of the student would be spent in solving problems. Below we have given a few examples of definitions, theorems, and problems that might come up in connection with each of the topics (1)-(9) described earlier. Some of the definitions have been omitted, owing to their complexity.

1. General Introduction to Transformations

Definition. If T is a transformation from S to S' (denoted by: $T: S \rightarrow S'$) and $A \subseteq S$, then

$$T(A) = \{T(x) \mid x \in A\}.$$

$T(A)$ is called the image of A under T.

Theorem. If $T: S \rightarrow S'$ and A and B are subsets of S, then

$$T(A \cup B) = T(A) \cup T(B) ,$$

i.e., T preserves unions,

Theorem. If $T: S \rightarrow S'$ and $U: S' \rightarrow S''$ are both one-to-one and onto, then

$$(U \circ T)^{-1} = T^{-1} \circ U^{-1} ,$$

i.e., the inverse of a composition is the composition of the inverses in reverse order.

Exercise. Suppose $S = S'$ = real numbers. Suppose $T: S \rightarrow S'$ is given by $T(x) = x^2 - 2x + 5$. If $A = [-4, 7]$, find $T(A)$.

Exercise. Suppose $T: S \rightarrow S'$ is one-to-one and onto. Prove that

$$(T^{-1})^{-1} = T .$$

2. Axial Reflections in the Plane.

Definition. If ℓ is a line in a Euclidean plane π, then $[\ell]$, the axial reflection of π into itself in the axis ℓ, is the unique transformation T defined by:

$$X \in \ell \Rightarrow T(X) = X \; ;$$

$$X \notin \ell \Rightarrow \ell \text{ is the perpendicular bisector of the segment with endpoints } X \text{ and } T(X).$$

Theorem. If ℓ and m are lines in π, then $[\ell]\,(m)$ is a line, i.e., axial reflections preserve lines.

Theorem. If ℓ and m are lines in π, then $[m] \circ [\ell] = [\ell] \circ [m]$ $\Longleftrightarrow \ell = m$ or $\ell \perp m$.

Theorem. If ℓ is a line in π and $\mathcal{E}$ is an ellipse with foci F_1 and F_2 and sum of focal radii = 2a, then $[\ell](\mathcal{E})$ is an ellipse with foci $[\ell](F_1)$ and $[\ell](F_2)$ and sum of focal radii = 2a.

Theorem. If ℓ is a line in π and A and B are any two points in π, then

$$d([\ell](A), [\ell](B)) = d(A,B) \; ,$$

i.e., axial reflections preserve distances.

Exercise. Prove that the tangents to a circle with center O from a point P outside the circle are equal. [Hint: Let $\ell = OP$, and consider $[\ell]$. Use the fact that $[\ell]$ preserves tangency and distances.]

Exercise. Prove that parallel lines cut off equal arcs on a circle. [Hint: Consider an axial reflection in the line joining the midpoints of the segments cut out by the circle on the two parallel lines.]

Exercise. Given a line ℓ and two points A and B on the same side of ℓ, find the point X on ℓ such that the sum $d(A,X) + d(X,B)$ is a minimum. [Hint: Consider $[\ell](A)$, then look at the line determined by $[\ell](A)$ and B.]

Exercise. Prove the reflection property of the ellipse (this is behind the principle of sound in elliptical-shaped rooms). [Hint: Use the preceding exercise and the definition of an ellipse.]

3. Inverses and Compositions of Axial Reflections. [Note: Since translations, central reflections and rotations can all be interpreted as compositions of axial reflections, and since these transformations have many interesting properties in their own right, this unit would probably be the most extensive one in the course. For that reason, the teacher should probably allot four or five weeks to cover the various subtopics included under (3)].

Definition. A translation of a Euclidean plane π into itself is a transformation T such that for all points X and Y in π, the quadrilateral $XT(X)T(Y)Y$ --in that order--is a parallelogram.

Theorem. If T is a transformation of π into itself, then T is a translation if and only if there exist lines ℓ and m such that

$$\ell \| m \quad \text{and} \quad T = [m] \circ [\ell] .$$

Theorem. If A and B are points in π, there is a unique translation T of π into itself such that $T(A) = B$.

Exercise. Prove that the inverse of a translation is a translation. [Hint: Use either the definition, or the fact that inverses of axial reflections are axial reflections.]

Exercise. Given circle $\mathcal{C}(O,r)$ and a segment $\overline{AB}$ in π with $d(A,B) \leq 2r$, construct the chords of $\mathcal{C}(O,r)$ parallel, and equal in length, to $\overline{AB}$. [Hint: Consider $T(\mathcal{C}(O,r))$, where T is the unique translation such that $T(A) = B$.]

Definition: A central reflection of a Euclidean plane π into itself is a transformation T such that for any points X and Y in π, the segments $XT(X)$ and $YT(Y)$ have the same midpoint.

Theorem. If T is a transformation of π into itself, then T is a central reflection if and only if there exist lines ℓ and m such that

$$\ell \perp m \quad \text{and} \quad T = [m] \circ [\ell] .$$

Theorem. If P is a point in π, there is a unique central reflection T of π into itself such that $T(P) = P$.

Theorem. The composition of two central reflections is a translation.

Theorem. Every translation can be expressed as the composition of two central reflections.

Exercise. Prove that the composition of two translations is a translation. [*Hint*: Express each translation as a composition of central reflections. Choose your central reflections wisely.]

Exercise. Prove that the diagonals of a parallelogram bisect each other. [*Hint*: Consider a central reflection in the midpoint of one diagonal.]

Exercise. Prove that the tangents to the endpoints of a diameter of an ellipse are parallel. [*Hint*: Consider a central reflection in the center of the ellipse. Use the fact that central reflections map lines onto parallel lines and preserve tangency.]

Definition. [Here one would give the definition of *rotation*]

Theorem. If T is a transformation of π into itself, then T is a rotation if and only if there exist lines ℓ and m such that

$$\ell \cap m \neq 0 \quad \text{and} \quad T = [m] \circ [\ell] .$$

Theorem. The composition of two rotations is either a rotation or a translation.

Exercise. Prove that the composition of a translation and a rotation is a rotation.

Exercise. Prove that the rotations and translations of a plane into itself form a group under the operation of composition. [Hint: Use the preceding exercise and theorem.]

Exercise. Prove that perpendicular lines through the center of a square divide the square into four congruent quadrilaterals. [Hint: If O is the center of the square, consider a clockwise rotation through 90° with center O.]

Exercise. Construct an equilateral triangle with its vertices one each on three given parallel lines. [Hint: Choose a point O arbitrarily on one of the three lines, and consider a rotation through 60° about O. Use the fact that a rotation with center at one of the vertices of an equilateral triangle and through 60° will map one of the sides of the triangle onto another side.]

4. General Notion of a Rigid Transformation

Definition. A rigid transformation of a Euclidean plane π into itself is a transformation T such that for all points X and Y in π,

$$d(T(X),\ T(Y)) = d(X,Y)\ ,$$

i.e., rigid transformations preserve distances.

Theorem. Rigid transformations map lines onto lines, segments onto segments, circles onto circles, polygons onto congruent polygons, etc.

Theorem. If $\triangle PQR$ and $\triangle P'Q'R'$ are two non-degenerate triangles such that $d(P,Q) = d(P',Q')$, $d(Q,R) = d(Q',R')$ and $d(P,R) = d(P',R')$, there is a unique rigid transformation T such that

$$T(P) = P', \quad T(Q) = Q' \quad \text{and} \quad T(R) = R' .$$

Theorem. If T and U are two rigid transformations which agree on three non-collinear points, then $T = U$.

Theorem. If T and U are rigid transformations, then T^{-1} and $U \circ T$ are rigid transformations.

Theorem. The set of all rigid transformations of a plane π into itself is a group under the operation $\circ$ of composition.

Definition. If $\mathcal{F}$ and $\mathcal{F}'$ are two figures in a plane π, then $\mathcal{F}'$ is said to be _congruent_ ($\mathcal{F}' \equiv \mathcal{F}$) to $\mathcal{F}$ if there exists a rigid transformation T such that $T(\mathcal{F}) = \mathcal{F}'$.

Exercise. Prove that rigid transformations map open half-planes onto open half-planes. [_Hint_: Use the fact that rigid transformations map lines onto lines.]

Exercise. Prove that if the vertices of two polygons $\mathcal{P}$ and $\mathcal{P}'$ -each with at least seven sides-can be matched with each other so that corresponding diagonals are congruent, then the polygons are congruent. [*Hint*: Consider vertices P_1, P_3 and P_5 of polygon $\mathcal{P}$, and vertices P_1', P_3' and P_5' in polygon $\mathcal{P}'$. Consider the unique rigid transformation T such that $T(P_1) = P_1'$, $T(P_3) = P_3'$ and $T(P_5) = P_5'$. Show that $T(\mathcal{P}) = \mathcal{P}'$.]

Exercise. If T is a rigid transformation and U is a central reflection with center P, show that

$$T \circ U \circ T^{-1}$$

is a central reflection with center $T(P)$.

Exercise. Prove that two ellipses are congruent if they have the same distance between their foci and the same sum of focal radii. [*Hint*: Use the theorem on the existence of rigid transformations.]

Exercise. Prove that any two lines are congruent.

Exercise. Prove that axial reflections, translations, central reflections and rotations are rigid transformations.

Exercise. Suppose T is a rigid transformation such that for any line ℓ in the plane π, $T(\ell) = \ell$ or $T(\ell) \parallel \ell$. Prove that T is a central reflection or a translation. [*Hint*. Choose two distinct points A and B. Consider the lines AB and $T(A)T(B)$.]

5. Rigid Transformations and the Fundamental Decomposition Theorem

Theorem. Any rigid transformation T of a plane π into itself is the composition of one, two, or three axial reflections.

Theorem. The composition of three axial reflections in parallel axes is an axial reflection.

Theorem. The composition of three axial reflections in concurrent axes is an axial reflection.

Definition. A glide reflection is a rigid transformation of the form $T \circ [\ell]$, where T is a translation in a direction parallel to ℓ.

Theorem. The composition of three axial reflections in axes which are neither parallel nor concurrent is a glide reflection.

Exercise. Prove that any rigid transformation is either a translation, rotation (includes central reflections), axial reflection, or a glide reflection.

Exercise. If $U_1 = T_1 \circ [\ell_1]$ and $U_2 = T_2 \circ [\ell_2]$ are two glide reflections with $\ell_1 \perp \ell_2$, what is $U_2 \circ U_1$? [Hint: Use the fact that the translation and axial reflection of a glide reflection commute.]

6. Rigid Transformations and Alternative Classification Schemes

Definition. [Here one would give the definition of an orientation on a line.]

Definition. [Here one would give the definition of an orientation in the plane.]

Definition. [Here one would give the definition of orientation-preserving and orientation-reversing rigid transformations.]

Theorem. Any axial reflection of a plane π into itself is orientation-reversing.

Definition. If $T: S \rightarrow S'$ and x is an element of S such that

$$T(x) = x ,$$

then x is a fixed point of the transformation T.

Theorem. If T is an orientation-preserving rigid transformation of π into itself different from the identity, then

i) T is a translation if T has no fixed points;

ii) T is a rotation if T has fixed points.

Exercise. Prove that the only orientation-reversing rigid transformation having no fixed points are glide reflections. [Hint: How many types of orientation-reversing rigid transformations are there?]

7. Rigid Transformations and the Theory of Paper-Folding

Exercise. By making various folds on a rectangular sheet of paper, construct a 60° angle.

Solution: Fold the paper in half lengthwise, obtaining the line AB, as in Figure (a):

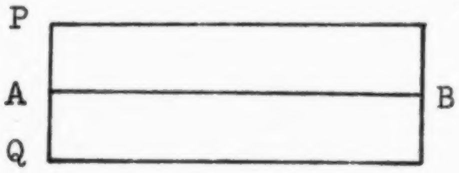


Figure (a)

Next, holding P fixed, lift the edge PQ until Q assumes a new position Q' on AB. Fold along the line PC, as shown in Figure (b)

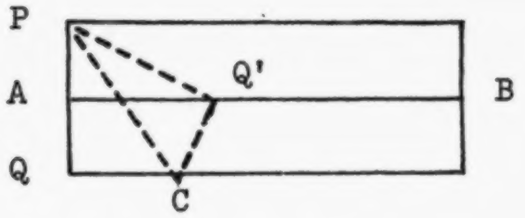


Figure (b)

Then $\sphericalangle QCP = 60°$. To see why this is so, let $\ell = AB$ and note that $[\ell](Q) = P$ and $[\ell](Q') = Q'$. Since axial reflections preserve distances, this means that $d(Q,Q') = d(P,Q')$. Hence $\triangle PQQ'$ is equilateral (why?), and from this it follows that $\sphericalangle QCP = 60°$.

There are many other problems similar to this one which can be interpreted in terms of axial reflections, and which can be solved using simple facts about axial reflections. The book by T. Sundara Row (see (15) in the bibliography) is an excellent source for such problems. Robert Yates in his book on geometric tools discusses three 'principles' or 'rules' for paper-folding; these are elementary folds from which more complicated folds can be made. These 'principles' can probably be used as a sort of axiomatic basis for the theory of paper-folding; moreover, they are amenable to interpretation within the language of geometric transformations using axial reflections. Details of this theory and its interpretation within the framework of rigid transformations remain to be worked out.

8. Rigid Transformations, Symmetry, and Groups.

Definition. If $\mathcal{F}$ is a figure in a Euclidean plane π and T is a rigid transformation of π into itself, then T is a *symmetry* of $\mathcal{F}$ if $T(\mathcal{F}) = \mathcal{F}$.

Theorem. Suppose $\mathcal{F}$ is a figure in π and $\mathcal{H}(\mathcal{F})$ is the set of all symmetries of $\mathcal{F}$. Then $\mathcal{H}(\mathcal{F})$ is a group under the operation of composition.

Definition. If $\mathcal{F}$ is a figure in π, then the set $\mathcal{H}(\mathcal{F})$ of all symmetries of $\mathcal{F}$ is called the *symmetry group* of $\mathcal{F}$.

Exercise. Define what would be meant by a *translational symmetry* of a figure $\mathcal{F}$.

Exercise. Show that if the symmetry group $\mathcal{H}(\mathcal{F})$ of a figure $\mathcal{F}$ contains a translation $T \neq I$, then the figure $\mathcal{F}$ is unbounded, i.e., cannot be contained within the interior of any circle.

Exercise. Show that the set of all *translational symmetries* of a figure $\mathcal{F}$ is a subgroup of the group $\mathcal{H}(\mathcal{F})$ of all symmetries of $\mathcal{F}$.

Exercise. Describe the symmetry group of a square in the plane.

9. Similarity Transformations

Definition. Suppose T is a transformation of π into itself and k is a positive number such that for all points X and Y in π,

$$d(T(X),T(Y)) = k \cdot d(X,Y) .$$

Then T is called a similarity transformation of π into itself.

Theorem. Similarity transformations map lines onto lines, segments onto segments, circles onto circles, polygons onto similar polygons, etc.

Theorem. If $\triangle PQR$ and $\triangle P'Q'R'$ are two non-degenerate triangles and k is a positive number such that $d(P'Q') = k \cdot d(P,Q)$, $d(Q'R') = k \cdot d(Q,R)$ and $d(P',R') = k \cdot d(P,R)$, there is a unique similarity transformation T such that

$$T(P) = P', \quad T(Q) = Q' \quad \text{and} \quad T(R) = R' .$$

Theorem. If T and U are two similarity transformations which agree on three non-collinear points, then $T = U$.

Theorem. If T and U are similarity transformations, then T^{-1} and $U \circ T$ are similarity transformations.

Theorem. The set of all similarity transformations of a plane π into itself is a group under the operation $\circ$ of composition.

Definition. If $\mathcal{F}$ and $\mathcal{F}'$ are two figures in a plane π, then $\mathcal{F}'$ is said to be similar ($\mathcal{F}' \sim \mathcal{F}$) to $\mathcal{F}$ if there exists a similarity transformation T such that $T(\mathcal{F}) = \mathcal{F}'$.

Definition. If T is a transformation of a plane π into itself, k is a positive number, and O is a fixed point of T such that for all points $X \neq O$ in π

i) $T(X)$ is on the ray $\overleftrightarrow{OX}$ and

ii) $d(O,T(X)) = k \cdot d(O,X)$,

then T is called a similitude of π into itself with center O.

Theorem. A similitude is a similarity transformation.

Theorem. If T is a similarity transformation of π into itself, there is a rigid transformation U and a similitude T_1 such that

$$T = T_1 \circ U .$$

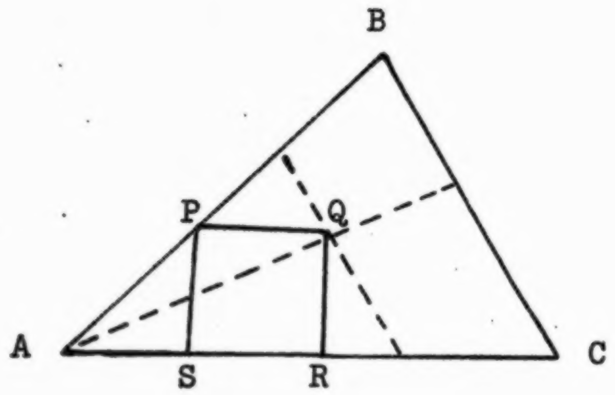
Exercise. Prove that similarity transformations map interiors of triangles onto interiors of triangles.

Exercise. Prove that any two lines are similar.

Exercise. Prove that two ellipses are similar if they have the same eccentricity.

Exercise. Prove that any two parabolas are similar.

Exercise. Inscribe a square in a triangle so that two of its vertices lie on one side of the triangle, and the remaining two vertices lie on the remaining sides of the triangle. [Hint: Construct a



square PQRS with P on AB and R and S on AC, as shown in the figure at the left. Consider an appropriate similitude with center at A.]

Exercise. Construct a triangle given two sides $\overline{AB}$ and $\overline{AC}$ and the median $\overline{AD}$ from A. [Hint. Lay off a segment $\overline{AB}$; find the locus of possible positions for C. Now consider an appropriate similitude with center B.]

III. Bibliography on Geometric Transformations

The following is a reasonably complete bibliography of works in English containing material on geometric transformations. Some of them, such as the books by Jeger and Yaglom, are translations from other languages.

1. Barry, E.H., *Introduction to Geometrical Transformations*, Prindle, Weber and Schmidt, Inc., Boston, 1966.
2. Benson, R.V., *Euclidean Geometry and Convexity*, McGraw-Hill, New York, 1966.
3. Coxeter, H.S.M., *Introduction to Geometry* , John Wiley, New York, 1961.
4. Coxeter, H.S.M., *Non-Euclidean Geometry* (5th ed.), University of Toronto, Toronto, 1961.
5. Eves, H., *A Survey of Geometry*--Vol. 1, Allyn and Bacon, Boston, 1963.
6. Fejes-Tóth, L. *Regular Figures*, Macmillan, New York, 1964.
7. Greenwood, D., *Mapping*, University of Chicago Press, Chicago, 1964.
8. Guggenheimer, H.W., *Plane Geometry and Its Groups*, Holden-Day, Inc. San Francisco, 1967.
9. Jeger, M., *Transformation Geometry*, John Wiley, New York, 1966.
10. Kelly, P., and Ladd, N., *Geometry*, Scott, Foresman and Co., Chicago, 1965.
11. Meschkowski, H., *Non-Euclidean Geometry*, Academic Press, New York, 1964.
12. Modenov, P.S., and Parkhomenko, A.S. *Geometric Transformations*--Volumes 1 and 2, Academic Press, New York, 1965.
13. Moise, E.E., *Elementary Geometry from an Advanced Standpoint*, Addison-Wesley, Reading, Mass., 1963.

14. Perfect, H., Topics in Geometry, Macmillan, New York, 1963.

15. Row, T.S., Geometric Exercises in Paper Folding, Dover Publications, and New York, 1966.

16. Schuster, S., College Geometry Project, Minnesota School Mathematics and Science Center, University of Minnesota, Minneapolis, 1966.

17. Shubnikov, A.V., and Belov, N.V., Colored Symmetry, Macmillan, New York, 1964.

18. Sohon, F.W., The Stereographic Projection, Chemical Publishing Co., Inc. Brooklyn, 1941.

19. Thomsen, G., The Treatment of Elementary Geometry by a Group Calculus. Math. Gazette 17, p. 232, 1933.

20. Tuller, A., A Modern Introduction to Geometries, D. Van Nostrand, Princeton, 1967.

21. Yaglom, I.M., Geometric Transformations, New Mathematical Library, Random House, New York, 1962.

The following two books, in German and French, treat the foundations of geometry from the point of view of geometric transformations.

22. Bachmann, F., Aufbau der Geometrie aus dem Spiegelungsbegriff, Springer, Berlin, 1959.

23. Delessert, A., Une Construction de la Géométrie Élémentaire Fondée sur la Notion de Réflexion. L'Enseignement Mathématique, Genéve, 1964.

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A few comments on some of these books may be helpful. The books by Jeger and Yaglom are organized around a series of problems; the appropriate theory of different types of transformations is developed as a means to solve these problems. Both are very rich in the extent and variety of geometry problems which can be solved using rigid and similarity transformations. The books by Barry, Modenov and Parkhomenko and Perfect are somewhat similarly organized, but move on to consider affine, projective, and inversive transformations. The books by Coxeter give a very good summary of the theory of transformations within a number of alternative geometries, though the exercise material is not very extensive. Fejes-Toth's, Shubnikov's and Belov's books concentrate on the various symmetry groups arising in connection with certain geometric figures. Groups are a predominant theme in Guggenheimer (8), Thomsen (19), Bachmann (22) and Delessert (23), but here attention shifts from symmetry groups to an axiomatic treatment of geometry using groups of transformations. Finally, Kelly's and Ladd's Geometry is a high school text containing much interesting material on transformations in the plane and applications of transformations to topics in area, volume, circles and spheres.

§2. The Multiple-choice Program

by

Peter Belew

The following section of this report deals with the implementation of the first two geometry lessons on the PDP-1 instructional computer system at Stanford. In these lessons, fixed diagrams are displayed on a cathode ray tube screen, and instruction and questions are either displayed as text on the screen under the diagrams or played by the Westinghouse audio system. The student then types answers on a keyboard under the screen. A language was devised in which lessons of this sort may be written, and an interpreter for lessons written in this language was written in TSA for execution on the PDP-1, and translations of the first two lessons were tested with this interpreter.

TSA and the Lesson Language

Several criteria were observed in the creation of this language. It was felt that it should be simple enough that persons untrained in programming could transcribe lessons written in a form similar to that of the geometry lessons into the source program language. In order that lessons can be arbitrarily long, it is necessary that the major part of a lesson program be kept in bulk storage, with only small segments being brought into core for execution at any one time. Since the bulk of a lesson program of this sort consists of text manipulation--with many long sections of text--and frequently used routines for handling display information and checking student responses, it was felt that an interpretive system, written in a language already providing procedures for text manipulation, as well as commands for handling pictorial displays and audio, would be appropriate.

Since a compiler and interpreter for TSA, a language which meets these requirements, is available on the PDP-1, it was decided to write the lesson interpreter in TSA. TSA is similar to Algol in the syntax of assignment statements, conditional statements and compound statements, allows procedure declarations (with calls-by-name of parameters being indicated by a "→" in the procedure call), but is radically different from Algol in typing of variables, assignment of storage to variables, and in the handling of arrays. Type assignment and storage allocation are dynamic; a variable name which has not yet appeared either in an array declaration or on the left side of an assignment statement is of type "nil"; a constant ('1', '37', 'true') or literal string ('"This is a string"') is of type 'integer', 'Boolean' or 'string'; certain reserved variables ('latency', 'timer', 'timeout') have a constant type ('integer' or 'boolean') as well as a fixed initial value; and a one-dimensional list which may be initialized to a prescribed length by an array declaration is of type 'array' with elements of type 'nil' until its elements have values (and hence types) assigned to them. The types of expressions allowed are arithmetic, with the usual integer arithmetic operators (as in Algol), Boolean (as in Algol) and string with concatenation (designated by '&'). When an assignment statement is executed, the TSA interpreter first examines the expression on the right side to see if it is the same type as the variable on the left side of the assignment statement (if they are arrays, the dimensions of the arrays and the types and dimensions of the elements must be checked), and if they are different, storage must be reassigned for the variable.

It then calculates the value of the expression on the right and stores it in the space allocated for the variable. An array element may have any value (and type) assigned to it by an assignment statement once the array has been created by declaration or assignment. For example,

```
Array A[3];
A[1] <- 3; A[2] <- "A";
B <- A;
B[3] <- A;
X <- B[3][2]
```

Gives X a value of "A" and type 'string'. A variable X may be given value and type 'nil' by the statement 'X <- Nil', thus freeing space in the free storage area of core.

Variables may also be assigned values by input procedures in one of two ways. First, data may be read from a text file on the disc storage unit. A number of pointers, identified by number, may be set by the 'data' statement to points in a file. Data may be read by a statement of the form 'X <- *N', which reads the item the nth pointer indicates and advances the pointer to the next item. A data item is either an integer, a string--which is either in the form of a sequence of characters containing neither spaces nor slashes ('/') and beginning with a letter, or any sequence containing no slashes which is delimited by a pair of slashes ('/dog/')--or a list (array) which is any sequence of data items enclosed in a parenthesis pair.

The other input mode accepts student replies from the keyboard. When the command 'RR' (response request) is given, a timer is started and the program begins to accept characters from the keyboard. If either a carriage-return is typed or a preset number of characters have been typed before the timer runs out, the characters are assembled into a string and assigned to the reserved variable 'answer', which is set to 'nil' by 'RR'. If time runs out, a flag is set. An answer is of type 'string' or 'nil'. However, if any string variable is of the form of a TSA data item, it may be interpreted as if it were a data item by the function 'convert'. For example,

```
X <- convert ("(/1 dog/ 1)")
```

has the result that X[1] equals "1 dog" and X[2] equals 1.

TSA has been developed at Stanford principally by Dow Brian and Brian Tolliver. No published description of the language exists at present, but this language and the entire Zeus system will be described in a forthcoming book by Professor Suppes of Stanford.

Description of the Lesson Language

Appendix II contains an example of a program in the lesson language which may be referred to here in connection with the following description. A statement of the language is a TSA data item of type 'array', i.e. one or more TSA data items enclosed in a parenthesis pair.

A question consists of a question label specifying the time and maximum number of errors allowed for a question, one or more caption and/or audio-message statements for displaying or playing the text of the question, and an answer list. When the interpreter encounters the question label, it sets pointers to indicate the point in the program where the question starts, so that the question can be repeated, if necessary. When the interpreter encounters the answer list, it types "answer!", initializes a timer, and begins to accept characters from the keyboard. When a carriage-return is typed, the interpreter decides whether the input string is a valid TSA data item, i.e. if parenthesis and string delimiters ("/") match. If it is valid, the interpreter then interprets the input either as a set of unordered elements, if this is specified at the beginning of the answer list, or as a string or integer as determined by the format of the input. What the student typed is then cleared from the screen, and the processed answer is typed in its place. The answer is then compared in turn with the items in the answer list. If a match is found, the validity (right or wrong) of the answer found is noted and any text or audio messages associated with the answer are displayed or played. If no match is found, "this answer is not on the list" is typed and the question is repeated, unless one item is of the form "(ELSE ⟨VALIDITY⟩)", in which case the appropriate validity is noted. If the validity is "WRONG", the audio part of the question is repeated, and the wrong answer score for the current section of the lesson is incremented by two. If the time allotted for the question is exceeded, "TIME OUT!" is typed, the wrong score for the current section is incremented by one, and the question is repeated, if the number of trials allotted for the question has not been exceeded. If the number of trials is exceeded by some combination of timeouts and wrong answers, "CALL TEACHER" is typed and the lesson is terminated. If an answer is right, "RIGHT!" is typed, the right score for the current section is incremented by two, and the interpreter proceeds to the statement following the "(END)" of the answer list.

Questions and other statements for displaying figures, and typing text and playing audio messages of the "lecture" part of a lesson are organized into sections for the purpose of checking the student and repeating parts which he seems to find difficult, or calling the teacher if the student makes an excessive number of errors. A section begins with a section label and ends with the statement prior to the next section label or the end of the current topic or lesson. Usually there are at the end of a section, and perhaps within a section, one or more action statements specifying action to be taken on the basis of a student's total wrong score for one or several sections: e.g. "(ACTION (1 4 5)(40 70) 4)" means that if the total wrong score for Sections 1, 4, and 5 is greater than 70, the teacher is to be called else if it is greater than 40, the interpreter should return to Section 4. When the interpreter encounters a section label, it sets a pointer to a place in the scoring table for the current topic (see next paragraph), indicating that right or wrong answers are to increment the scores at this point in the table, and sets these scores to zero.

Sections are organized into larger units called topics, essentially corresponding to the topics in the original textbook lessons. A topic begins with a topic label-- "(TOPIC N)" -- and ends with "(FINIS)." When the interpreter encounters a topic label, a section directory (the array "LOCSEC") and the scoring table (the arrays "RIGHT" and "WRONG") are created. LOCSEC is generated either by means of a declaration-- "(LOCSEC ⟨TABLE OF PAGE AND LINE NUMBERS⟩)"--or, in the absence of a declaration, by searching through the topic for section labels. It is necessary to create these tables anew whenever a topic is begun because of the limited storage space available with the TSA system; because of this limitation, no jumps are possible from within one topic to a section in another.

An entire lesson is terminated by the statement "(STOP)." This stops the data scan and returns the interpreter program to its start for running another program.

Timing Considerations and Other Problems

There are two principal factors leading to unsatisfactory execution speed of the lessons: nonoverlapping audio messages and limited storage and buffer space in the TSA system. Unfortunately, the TSA audio commands are nonoverlapping, that is, after a program has requested the audio control to play an audio message, it must wait until the message has been completed before the program can proceed. It would be most desirable to allow a text to be typed on the screen simultaneously with the playing of the same message on the audio unit. This would help cover up delays in typing lengthy texts since the spoken message would be played at a speed comparable with the speed at which text is typed under moderate system load.

The other problem, limited space for the input buffer and free storage (of variables in the TSA interpreter), is somewhat more important. The present size of the TSA data buffers is on the order of 60 characters per data pointer (only two pointers are allowed at present) so that frequent disc accesses are required when reading data--in this case, when the lesson interpreter is getting statements from the lesson file. When a number of users are making access to the disc, they frequently have to queue up for their data transfers, which can drastically slow down the access time for a given user, since it may be necessary for the heads on his disc module to move from track to track to reach one or more other user's files. The limitation on free storage also slows down the functioning of the interpreter. When an answer is compared with the items in the answer list of a question, it is always necessary to convert the raw data items into TSA lists before checking them, and frequently necessary to make more than one disc access in order to fill the buffer--a relatively slow process in any event. If it were possible to read the entire answer list as one data item, much of the analysis of the list could be done by the time the student begins to react to the question; moreover, the list could be organized much more efficiently. For example, an answer list of a form such as

```
(ANSWERS (YES RIGHT)(NO WRONG)(ELSE WRONG))
```

or even the present form

```
(ANSWERS)(YES RIGHT)(NO WRONG)(ELSE WRONG)(END)
```

could very quickly be converted into a table

```
((YES RIGHT)
 (NO WRONG)
 (ELSE WRONG))
```

when the interpreter reads the list--while the student is reading or listening to the question--and the comparison of the student's response with the choices in the table can be done very rapidly in core.

In addition to the difficulties leading to slow running of the lessons, the limitation on space for a user's display buffer has been felt. In this system, each user has a 330 word buffer available to him in core into which he may load instructions for generating his display. The instructions are either text words of three characters, gross positioning instructions, instructions for drawing short line segments (which may be invisible, or have only the end points brightened, if desired) or control words for choosing one of the above modes or specifying the brightness of the display. About every 200 milliseconds the PDP-1 transfers all 12 user's display instructions--up to about 4000 words--word by word to the Philco display control which then translates the instructions into beam intensities and positions.

The diagrams for the geometry lessons were translated into display instructions by means of two programs: a figure generating program (known as "doodle") and a display editor. The doodle program translates simple instructions typed on the keyboard describing coordinates for line segments, circles, and ellipses into display instructions; circles and ellipses are constructed out of a large number of short line segments. After a figure is drawn, its coordinates may be changed by commands typed on the keyboard. During this process the figures generated appear on the Philco screen along with a display word count. When a display has been completed, it may be written on a disc file by typing the appropriate command. The display editor, on the other hand, is used for word-by-word modification of a preexisting display, including inserting text which uses special characters not in the Philco character set, which have been previously constructed out of line segments by the display editor. A typical use of the display editor would be to delete alternate segments in a circle to create a dotted line effect. Since there are at present only 330 words of display buffer available for each use and figures such as circles and other curves must be made up of a large number of short line segments, there is a severe limitation on the complexity of the figures which may be displayed. However, changes are contemplated in the system which will enable users to tailor their free display buffer size to their needs.

§3. The Independent Student Program

by

John Lennie

One of the goals of the project was to develop a program in which the lessons would be specified more loosely than in the Multiple-choice Program. The lessons would consist of explanatory material, sequences of problems, branch criteria, etc. The student would be left free to invent his own solution to a problem rather than to choose one from a fixed set of alternatives. At any time he would have a set of theorems and constructions at his disposal and the set would grow as he assimilated new material.

It proved impossible to realize this goal, partly for practical reasons having to do with program complexity but also because of logical difficulties in programming the understanding necessary to bridge the rather large steps that are almost unavoidable in geometric proofs. Therefore the program described below is not completely written but the description incorporates our best thoughts about the organization of the unfinished parts. The program is written in assembly language which is not a good medium for communication, unlike TSA. The choice was forced by the need for efficient execution and large data structures.

Configuration

We restrict the program to handle one student: extending it to handle several students is not logically very difficult but absorbs an amount of computer time proportional to the number of students.

The student sits at a Philco console, sees the screen in front of him and uses the keyboard and light-pen for his input. The audio unit might be used for fixed text if desired.

Control of the Lesson

The program reads the lesson from a file on the disk. Each section of the lesson may simply be explanatory and require no reaction from the student-text on the screen or an audio message - or else a problem that does require a reaction. The sequencing of sections can be arranged as in the Multiple-choice Program. Evaluating right answers to problems is one thing (and is explained below) but wrong answers are more difficult to handle because anything is possible. Unfortunately, we cannot hope at this point to follow a misguided proof and point out where it went wrong, as a teacher could. Instead, we will have to use a broad criterion like a time-limit and possibly some suggestions which might or might not be relevant when the time limit is exceeded. It might also be worthwhile to put some limit on the activity of the student, measured by the number of statements he makes.

Input

When a problem is specified in the lesson it will typically consist of a text to be displayed and a set of statements to set up the initial conditions. The program will translate these statements and then switch its attention to the student who will type in more statements from the keyboard. The last of his statements should be 'Q.E.D.' or 'Q.E.F.' (or whatever equivalent messages are convenient). The program can then check that the conditions which the problem called for have in fact been established.

The statements are part of an incremental programming language which manipulates the computer's model of the problem (which, needless to say, is not the diagram which the student sees but something more fundamental from which the diagram is derived). The language is incremental in the sense that a theorem which has been proved becomes available as a new statement.

Because the set of statements is changing we can imagine that the student is given a list of the theorems prepared on a teletype before the lesson starts. New theorems acquired during a lesson could be optionally displayed, presented on a teletype, or simply added to the end of the prepared list to be used only when they have been proved. The special statement which initiates a new definition will only be read from the lesson file.

The light-pen may be used in two ways, first to put a point in a particular position in the diagram (we have to treat quite differently the two cases where the point is not related to anything and, say, where it is put on a line) and second to indicate an object (instead of typing 'circle ABC' the student could be allowed to pick it out with the light-pen). The second function could be dispensed with but it happens that the hardware makes it very easy to answer provided the light-pen is not used in the ambiguous region where several objects intersect.

It may be noted here that the light-pen is not intended to be used to move a point and reconstruct the diagram entirely so that it again satisfies the logical relations involving that point. (This was done in that innovating graphics program, Sketchpad [5].) However restructuring of certain kinds is implied by transformations--in fact this is one of the two places where the specifically transformational approach has programming implications.

We will normally require unambiguous identification of the components of a diagram, e.g. a new point must be given a position. Each component is assumed to have a name which may be arbitrary (as when a point is called 'M' rather than 'N') or constructive ('the intersection of AB and XY'). The latter way may introduce ambiguities because, in the above case, the lines may be parallel or, if one component is a circle, the intersection may not be unique. Of course it is possible to associate an arbitrary name with a constructive one, i.e. there is a statement to achieve the effect, 'let M be the intersection of AB and XY.'

A statement which is the application of a theorem is in two parts, one to identify the theorem and the second to make the appropriate parameter substitutions. The listing of theorems that the student has will provide him with a concise form for doing this. The effect of such a statement is that the program checks that the conditions which establish the theorem are true under the given parameter substitution. If the conditions are not true it is reasonable to allow the student to start again or to proceed to prove the theorem. That is, the proof checking program is recursive.

Of course, the application of a theorem is quite different from the proof of a theorem. In the former case we merely have to confirm that we have a correct substitution instance. As we said the creation of new theorems should be the prerogative of the lesson. However there is one important case where the student needs to create his own temporary theorems. This is in a problem with symmetry where he will prove a fact for part of the figure then, applying symmetry, say that 'similarly' the fact holds for the other part or parts. This is one of the logical difficulties mentioned earlier.

The other logical difficulty is deciding when a sufficient set of cases has been proved to establish a theorem in full generality. The problem may be illustrated by considering the 'proof' that every triangle is equilateral which involves taking impossible but plausible cases and turning a blind eye to the actual ones. In this example the variables which cover the cases have been given the wrong range of values but in a particular problem it is not easy to decide what are the significant variables nor what are their ranges. Many of the theorems that were intended to be proved do, in fact, require a non-trivial analysis into cases.

Besides the statements which refer to a diagram the language must include algebraic statements so that group-theoretic proofs about transformations can be made.

So far we have only said what the context and effect of the statements should be and said nothing about syntax. The language chosen is based on Strachey's macrogenerator [4] and assumes prefix rather than infix operators, which is rather inconvenient. It would, perhaps, have been better to have developed a more sophisticated format matching process, such as that used in LEAP [6].

While the student is preparing a statement the program effectively acts like a text editor with the additions necessary to handle the functions of the light-pen. When the student types a terminator the program translates the statement with the macrogenerator and calls another program to interpret the output. This new program maintains the model which includes both the logical information that is built up by the statements and the coordinate information that is needed to construct the diagram.

The Data Structure

The choice of a good internal model was a major difficulty in designing the program. On the one hand the model could be constructed in LISP [2] with all the advantages for experimenting with different organizations that come with programming in a higher level language. On the other hand, a special representation could be developed that would be concise and efficient to use. Sketchpad [5] has had such an effect on subsequent graphic programs that almost all of them have used a data structure which is an elaboration of its data structure (an excellent and up-to-date survey of various graphics-oriented data structures is [1]). There is no disagreement about what has to be stored but a fair amount of variation in the way data is accessed. The approach we finally decided on breaks away from the block-and-ring inheritance of Sketchpad and was influenced by Feldman and Rovner's associative memory scheme [3].

We are concerned with certain objects, their parameters and relations. Thus points, segments, rays, lines, triangles, circles and angles are objects, i.e. the nouns in the language; coordinates, orientation, length, etc., are parameters, i.e. ordered sets of scalar values; incidence, congruence, etc. are relations, i.e. the predicates.

We suppose each object has a name, which may be arbitrary or constructive as already explained. We represent each object by a pair of words which are (1) a pointer to the name of the object and (2) its type (segment, angle or whatever). The objects are kept in core permanently. The names are stored in one open-ended area of core and have 'subroutine' references so that a constructed name will have a pointer to the name of its component rather than a copy of the name (this is to simplify name changes, as by a 'let' statement). The parameters and relations are to be stored in blocks. We have the problem encountered with any list-processing language of representing an unknown amount of data and here we choose to do it by being very lavish with space. Because we have a good idea of the complexity that will be involved (i.e. the amount of detail in a diagram) we can allocate fixed-size areas for the different relations. The intent then is to hold most of this data on the disk. A relation in connection with a type will be used to construct an address of a block on the disc, e.g. the block of points incident with circles. When the block arrives we can use the name of the particular object (or an index of the name) to pick out the part of the block where its values are. Blocks on the disk are 174 words long so if we set aside one block for points incident on circles and allow 10 points per circle (which is a maximum for any circle rather than an average), we can represent up to 17 circles. The value of such a relation is a set and the value of an element of the set is simply a pointer to the object. When the value is a parameter it is symbolic rather than numeric. Thus the block containing segment lengths does not contain the actual lengths calculated from the coordinates of the end-points but symbolic values which may be compared for precise equality (as well as a numeric part to represent twice the value, half that value, etc.). Relations often have inverses, so that a point on a circle implies a circle through a point. We can represent this fact twice, but this violates the moral principle that representation should be unique, or else we can interpret one relation in terms of the other, e.g. as a search through the sets of points incident on all circles to find the set of all circles incident on one point. Of course, it is possible with parameters (whose number is fixed and not variable like the number of points on a circle) to hold several per block but it is fairly important not to distribute one parameter over several blocks.

Some blocks could be held permanently in core, notably the ones which contain the coordinate information for constructing the diagram. Note that with this kind of organization it is likely that a sequence of requests for blocks will refer to relatively few blocks. For example, if we are proving the congruence of two triangles, at worst we will want the following; the triangle-congruence block, the segments-in-a-triangle block, the angles-in-a-triangle block, segment-length and angle-magnitude. To evaluate the effort it is well to remember the considerable awkwardness in using ring structures for searching, with constant references from the ring item to see what object is currently being threaded. With room in core for 8 or 12 blocks there is some possibility of minimizing block transfers by writing out rarely used ones before more frequently used ones.

It should be emphasized that this scheme is not suitable for all graphics applications. It is feasible here because (1) we are only dealing with small sets--up to about 12 members and (2) the disk block size is appropriate.

Output

The final stage is construction of the diagram. Each object is processed by a routine appropriate for its type and as the display buffer is built up a pointer from its partitions to the objects is set up so that the light-pen can be used to pick out objects (one of the values made available by the light-pen interrupt is the relative address in the buffer).

In the case of a badly constructed statement the macrogenerator will return an appropriate error comment and allow the student to edit his statement or start a new one.

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[3] Rovner, P.D., and J.A. Feldman, An Associative Processing System for Conventional Digital Computers, Lincoln Laboratory Technical Note 1967-19 (April 1967).

[4] Strachey, C., A general purpose macrogenerator, Computer Journal 8 (1965) 225-241.

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PART III. CONCLUSION AND RECOMMENDATIONS

§1. Evaluation of Project

In looking back over the two years of the project the main conclusion of the Project Director is that the work should have been carried out on a larger scale. There was only one full-time staff member (M.H. Millar, Research Associate) and there should have been others. In particular, the Project Director should have taken at least one year off from his University duties to devote full-time to the project. Also, we should have had the advice of another college mathematician and of a high-school teacher - at least half-time each.

Another conclusion about how the project should have been organized concerns the material. The course material on transformations should have been taught in a high-school on an experimental basis. The purpose of teaching the course would have been not so much for evaluation but rather for actual development of the material at the right level. There is a lot of interest in transformations now and clearly the ideas can be integrated with other geometric ideas all the way from K to 12. But at the present time a course could only be given for one semester for special students in grade 12. Usually such students want calculus hoping to get a head start on college. That seems to us to be a serious mistake, though the feeling is understandable. More work in algebra and more work with abstract ideas such as those which can be illustrated by geometric transformations seems much more to the point from the college viewpoint.

The third major conclusion we reached was that the problem of programming the lessons for the machine was too hard for us. We had thought that we could start organizing the lessons as programs at once, but there were too many technical difficulties - especially those concerned with display of pictures. We should have concentrated on simply organizing the material for the classroom as our major activity. The machine programmers (in our case, Computer Science graduate students with many excellent ideas and much enthusiasm) could have been observers and experimenters. The responsibility of making complete lessons was too discouraging. We should have tried out only fragments of lessons or programs for special purposes like figure construction on the machine. There are certainly enough conceptual and organizational problems just at that level of machine use, as we found out. Any construction of complete lessons really requires a larger staff and some full-time programmers.

§2. Suggestions for Future Work

Aside from the obvious suggestion that the course should be taught and the material should be integrated with geometry at lower grade levels, the main area for future work seems to us to be on the machine. This is a very broad problem. There is considerable resistence to machine use in the classroom, but just considering this one rather limited area of geometry we have the following inescapable conclusion: graphical display of geometric figures controlled by computers is by far the most flexible method of illustration available today. This potential of making complicated constructions immediately visible and instantly adjustable and transformable needs to be put in the classroom. It should revolutionize geometry and create interest and stimulate curiosity far beyond the actual material treated. No doubt this will come soon in view of the way that various media are moving into the education business; however, we hope that research on these problems will not be left solely to the industry. In particular effective use of this display potential needs clear conceptual thinking and serious consideration of content of material.

Another area of machine use that seems to require further development concerns the interaction between the student and the machine. At the present time the machine can tell fairly well if the student is right, but if he is going wrong it is difficult to anticipate all the mistakes and put him back on the right track. Other difficulties in value problems of proof checking: the machine is good at details but what about arguments by symmetry and similarity - or worse by analogy.

We have to be careful that machine controlled lessons are not stifling good ideas that may be too "original" to have been anticipated when the program was written. Also, a fascinating area of research concerns data structures for representing the essential aspects of the material in the machine. The simple question "What is a geometric figure?" involves a considerable conceptual problem which is only agravated when one tries to produce machine representations. There have been many special purpose suggestions, but much remains to be done.

APPENDIX I

AUTHOR'S MASTER TEXT

Two Lessons on Transformations

by

Michael H. Millar

How to Read the Lessons

There is the same distinction to be made between the description of a lesson and its use as there is between the text of a computer program and its execution. In execution there are atomic events and criteria which decide their sequence, and the latter must be explained or understood along with the description of the events. Strictly branching in the lessons does require the use of labels just as in a program but as it is a particularly simple kind of branching - repeating a question until the right answer is given or interpolating material on certain conditions - it can be inferred from the sense and labels are not used. However when the lessons are transcribed (see Appendix II) the branching must be made explicit. Usually the sequence of execution follows the sequence of description except that there may be some overlapping of audio events and display events and some display events may persist while others come and go (as with a diagram and a series of problems which refer to it). The condition for one event to follow another is the elapse of a certain time for explanatory material and the right response from the student for a problem.

(The conventions for wrong responses and inattention are described along with other remarks on timing and the execution of parallel audio and display commands in the description of the Multiple-Choice Program.)

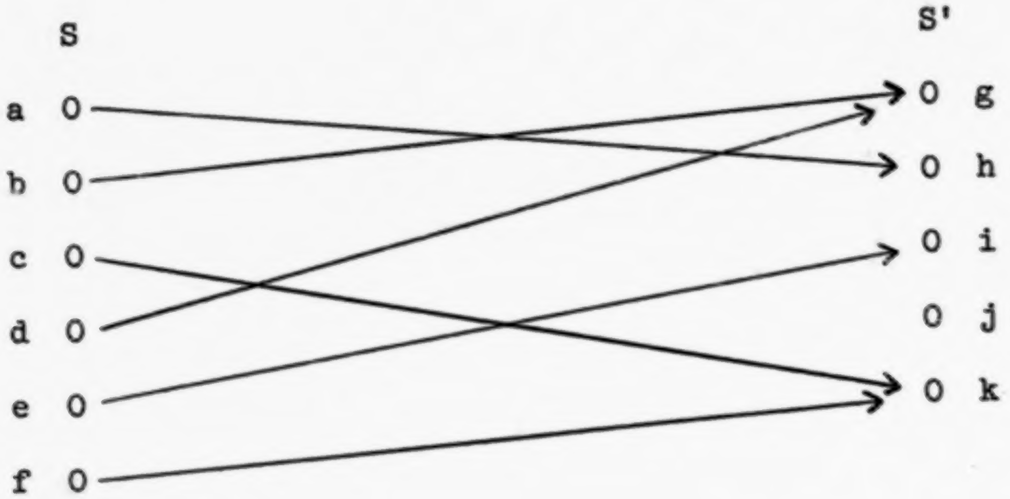
After each question is a comment starting Responses: followed by a list of alternatives separated by semicolons. These are displayed on the scope and are numbered so that the student usually identifies his response by typing a number (or sequence of numbers) rather than the text of the response. These numbers are, of course, omitted in the description and the correct alternative, or alternatives, is underlined. (When the correct response consists of several parts these parts and only these must be given, though the order is arbitrary.)

LESSON 1

[Audio] Topic 1. Definition of a transformation: Its well-defined character

[Diagram below appears and is to remain through (14)]

[Audio:] Look at the figure shown on the scope. This figure gives a schematic illustration of what is called a transformation from a set S into a set S'. By definition, a transformation from a set S to a set S' is a correspondence of the elements of one set (S) with some or all of the elements of a second set (S') in such a way that no element of S is made to correspond to more than one element of S'. The arrows are meant to tell us exactly how the elements of S are associated, or made to correspond, with elements of S'.



[Questions (1)-(14) are Audio.]

1. Two elements of S are a and d. What are the remaining elements of S? [Responses: 1: [b, c]; 2: [g, h]; 3: [b, c, e, f]; 4: [b, c, e, f, g, h, i, j, k]]

2. How many elements are there in the set S'? [Responses: 4, 5, 6]

[If 4: Audio:] No, for j is still an element of S' even though nothing in S corresponds to it. Try again.

[If 6: Audio:] No, I think you are confusing the set S' with the set S, which does have six elements. Try again.

3. The element a of S corresponds to the element h of S'.
    - i) What does the element b of S correspond with? [Responses: g; h; i; j; k]
    - ii) What does the element c of S correspond with? [Responses: f; i; k]
    - iii) What does the element d of S correspond with? [Responses: a; k; the same element as that to which b corresponds]
4. Elements b and d both correspond to which element in S'? [Responses: g; h; i; j; k]
5. Is there any other element in S' to which two or more elements of S correspond? [Responses: Yes; No]
6. What is this element? [Responses: g; j; k]
7. What are the elements of S corresponding to this element k in S'? [Responses: b and d; b and f; c and f; c and d]
8. Is there any element of S corresponding to two elements of S'? [Responses: Yes; No]

[If Yes: Audio:] You had better look again. Each element of S corresponds to exactly one element of S', though there are different elements of S that correspond to the same element of S'. So, try again.

9. Since no element of S corresponds to more than one element in S', what does the figure on the scope illustrate? [*Responses*: A correspondence between all the elements of S and all the elements of S'; a transformation from the set S' into the set S; a transformation from the set S to all of the elements of S'; *a transformation from set S to set S'*]

10. In the transformation illustrated on the scope, are there any elements of S' that are not the correspondents of any elements of S? [*Responses*: *Yes*; No]

11. What are these elements? [*Responses*: g; i; *j*; k; g and k]

12. From the example shown on the scope, would you say that if we have a transformation from a set S to a set S', then every element of S' must be the correspondent of some element of S? [*Responses*: Yes; *No*]

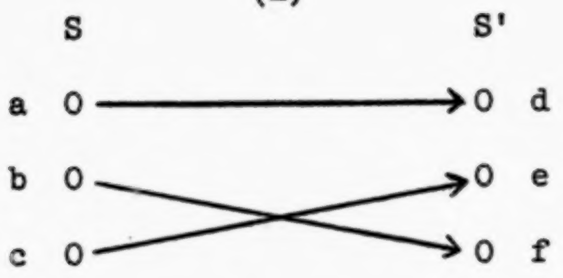
13. And if we have a transformation from S to S', must elements of S' be the correspondents of exactly one element of S? [*Responses*: Yes; *No*]

14. To be a *transformation*, however, each element of S must correspond to how many elements of S'? [*Responses*: At least one; sometimes none; sometimes two; *exactly one*]

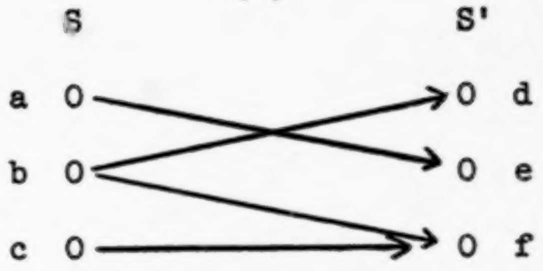
[Diagrams below are to appear now, and will remain through (5)]

[*Audio*:] Let's try a few problems now to see whether the idea of a transformation is clear to you. Look at the three diagrams shown on the scope.

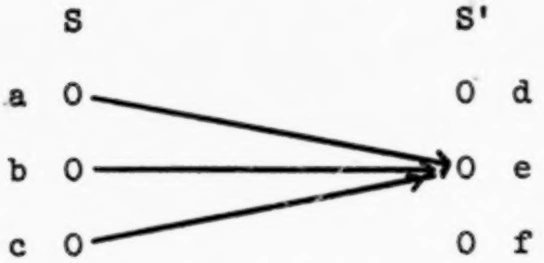
(1)



(2)



(3)



[Questions (1)-(4) are Audio; directive in (5) is Audio.]

1. Which of these diagrams illustrate a transformation from S to S'? [Responses: 1, 2, 3; 1 and 2; 2 and 3; 1 and 3; 1]

2. Diagram 2 does not illustrate a transformation since there is an element in S which corresponds to more than one element in S'. Which element is this? [Responses: a; b; c; d; f]

3. In Diagram 2, what are the two elements in S' to which the element b corresponds? [Responses: d and e; e and f; d and f]

4. For a correspondence from a set S to a set S' to be a transformation, however, each element of S must correspond to exactly how many elements in S'? [Responses: one; two; the element e of S']

5. Answer the following questions. [Questions here will appear in written form below the diagrams--no audio time used.]

i) In Diagram 1, are there any elements of S' that are not correspondents of elements of S? [Responses: Yes, No]

ii) How could Diagram 2 be altered to illustrate a transformation? [Responses: Connect a to d by an arrow; remove one of arrows pointing to e; remove the arrow pointing to d]

iii) In Diagram 3, how many elements of S correspond to e? [Responses: 0; 1; 2; 3]

iv) In Diagram 3, how many elements of S correspond to f? [Responses: 0; 1; 2; 3]

v) How many elements of the set S' in Diagram 3 are correspondents of elements of S? [Responses: 0; 1; 2; 3]

vi) Does Diagram 3 illustrate a transformation? [Responses: Yes; No]

> [If No: Audio:] Think about this again. Remember that our definition of a transformation requires only that each element of S corresponds to exactly one element of S'. And that is true here. Try the following questions now to check this point: [Questions will again appear in written form below the diagrams.]

vi-1) In Diagram 3, what is the one element of S' to which a corresponds? [Responses: e; f; g]

vi-2) What is the one element of S' to which b corresponds? [Responses: e; f; g]

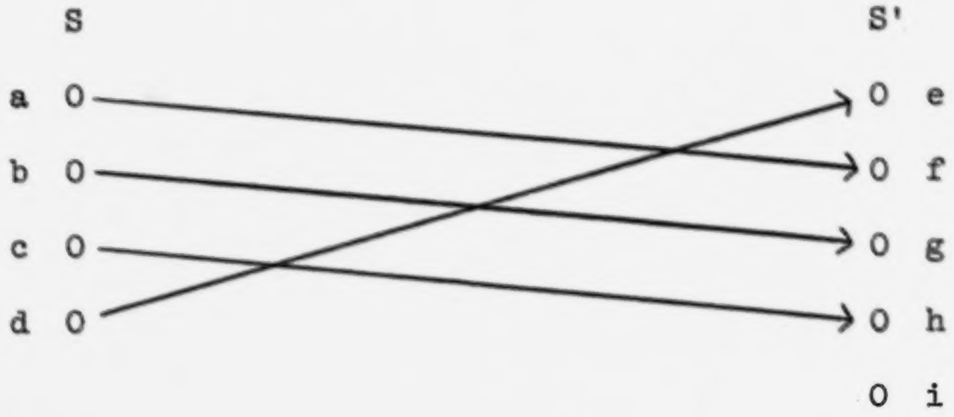
vi-3) What is the one element of S' to which c corresponds? [Responses: e; f; g]

[Audio:] You see, then, each element of S corresponds to exactly one element of S'. This element happens to be e in all cases, but that is irrelevant when we are considering only whether the correspondence illustrated in Diagram 3 is a transformation.

(Audio) Topic 2. *Domain and range of a transformation*

[Diagram below appears and is to remain through (15)]

[*Audio*:] When we talk about a transformation, we are usually interested in knowing what the *domain* and the *range* of the transformation are. Let's look at the transformation illustrated on the scope.



[*Audio*:] We say *by definition* that the *domain* of a transformation is the set of all elements of the first set that correspond to some element in the second set. In terms of the diagrams we have been using, an object would be an element of the *domain* of a transformation when there is an arrow *leading from* this element.

[Questions (1)-(15) are *Audio*.]

1. What is the domain of the transformation shown on the scope? [*Responses*: *S*; S'; {e, f, g, i}]
2. Is there an arrow leading from the element c in S? [*Responses*: *Yes*; No]
3. Is c an element of the domain of this transformation? [*Responses*: *Yes*; No]
4. Is there an arrow leading from the element e? [*Responses*: Yes; *No*]

5. Is e an element of the domain of this transformation? [*Responses*: Yes; *No*]

6. Name four elements besides e that are not in the domain of this transformation. [*Responses*: a; b; c; d; e; *f*; *g*; *h*; *i*]

7. List all the elements that comprise the domain of this transformation. [Responses: *a*; *b*; *c*; *d*; e; f; g; h;

[*Audio*:] The *range* of a transformation, on the other hand, is the set of all elements of the second set that are correspondents of one or more elements of the first set. In terms of our schematic diagrams, an object would be an element of the *range* of a transformation when there is an arrow *leading* *to* this element.

8. All elements of the range of the transformation shown on the scope belong to which set, S or S'? [*Responses*: S; *S'*]

9. Is there an arrow leading to the element e in S'? [*Responses*: *Yes*; No]

10. Is e in the range of this transformation? [*Responses*: *Yes*; No]

11. Is d in the range of this transformation? [*Responses*: Yes; *No*]

12. How many arrows lead to i? [*Responses*: *0*; 1; 2; 3]

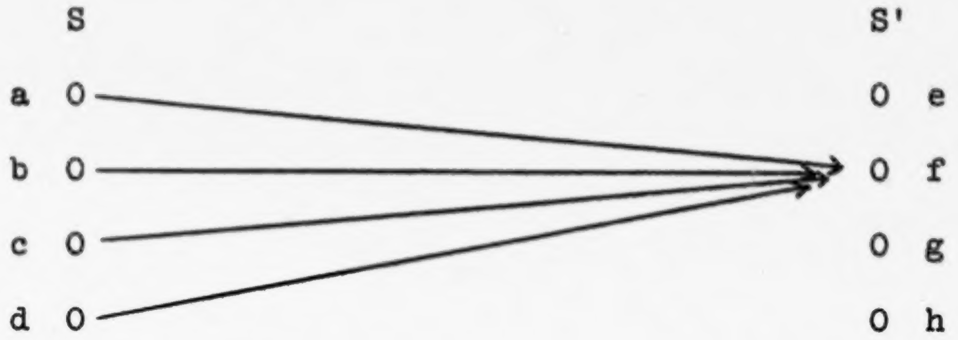
13. Is i in the range of this transformation? [*Responses*: Yes; *No*]

14. What is the range of this transformation? [*Responses*: S; S'; {h}; *none of these*]

15. List the elements in the range of this transformation. [*Responses*: a; b; c; d; *e*; *f*; *g*; *h*; i]

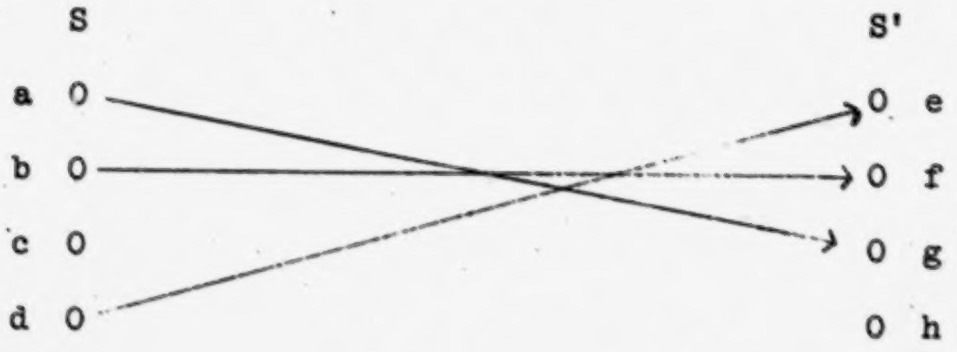
[Audio:] O.K., I will now keep quiet for a while. Let's see you run through the following problems. Just remember our definitions of transformation, domain, and range. [Questions here will appear in written form below the following diagram (which is to remain through (6))-- no audio time used.]

[Diagram below appears and is to remain through (6).]



1. What is the domain of this transformation: [Responses: {a, b, c, d}; {e, f, g, h}; {a, b, c, d, f}; {a, b, c, d, e, f, g, h}]
2. What is the range: [Responses: {e}; {f}; {g}; S; S']
3. Is there an arrow leading to g? [Responses: Yes; No]
4. Is g in the range? [Responses: Yes; No]
5. Use the typewriter to indicate the elements of S' that are not in the range of this transformation. [Responses: 1. a; 2. b; 3. c; 4. d; 5. e; 6. f; 7. g; 8. h.] [Student should type (5), (7), (8).]
6. If we have a transformation from a set S to a set S', does the range always have to be the set S'? [Responses: Yes; No]

[Diagram below appears and is to remain through (15). Questions will appear in written form below the diagram.]



1. List the elements in the set S. [*Responses*: a; b; c; d; e; f; g; h]

2. How many arrows lead from a? [*Responses*: 0; *1*; 2]
3. How many arrows lead from b? [*Responses*: 0; *1*; 2; 3]
4. How many arrows lead from c? [*Responses*: *0*; 1; 2; 3]
5. How many arrows lead from d? [*Responses*: 0; *1*; 2; 3]
6. List the elements of set S that have exactly one arrow leading from the element. [*Responses*: a; b; c; d; e; f; g; h]

7. Consider the set {a, b, d}. Do we have a transformation from this set to S'? [*Responses*: *Yes*; No]
8. What is the domain of this transformation? [*Responses*: S; {a, b, c}; *{a, b, d}*; {c}; S']
9. Is the domain of this transformation the same as the set S? [*Responses*: Yes; *No*]
10. Do we have a transformation from the set S to the set S'? [*Responses*: Yes; *No*]
11. But do we have a transformation from a *subset* of S to S'? [*Responses*: *Yes*; No]

12. Tell me again what this subset is. [Responses: S; {b, c, d}; {c}; {a, b, d}; the empty set]

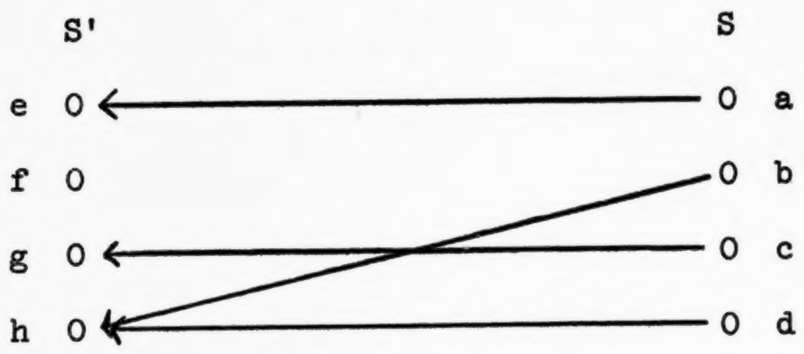
[Audio:] In nearly all of the situations we will be dealing with, we will have two sets S and S' specified and a transformation from S to S', that is, from all of the elements of S to elements of S'. But there are a few situations--such as the one shown on the scope--in which we will have a transformation only from some proper subset of S to the set S'. We will see a few further examples of this in a little while.

13. List the elements in the range of the transformation with domain {a, b, d}. [Responses: a; b; c; d; e; f; g; h]

14. Is the range all of S'? [Responses: Yes; No]

15. Which elements of S' are not in the range of this transformation? [Responses: c; f; g; h; c and h]

[Diagram below is to appear and remain through (8). Questions will appear in written form below the diagram]



1. List the elements in set S. [Responses: a; b; c; d; e; f; g; h]

2. Is g in S? [Responses: Yes; No]

3. Is there exactly one arrow leading from b? [Responses: Yes; No]

4. Is there exactly one arrow leading from h? [Responses: Yes; No]

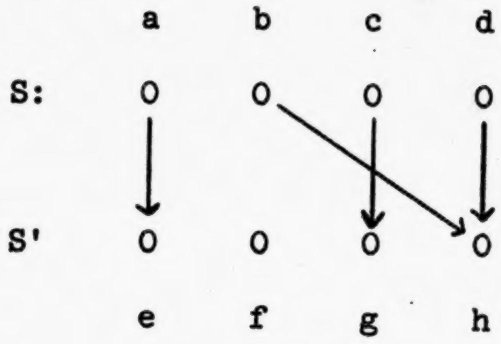
5. Do we have a transformation illustrated by the diagram? [Responses: Yes; No]

6. What is the domain of this tranformation? [Responses: S; S'; {e, g, h}]

7. What is the range of this tranformation? [Responses: S; S'; {e, g, h}; {f}]

8. Is this a transformation from S to S', from S' to S, or from {e, g, h} to S? [Responses: from S to S'; from S' to S; from {e, g, h} to S]

[Diagram below is to appear and remain through (8). Questions will appear in written form below the diagram]



1. List the elements in set S'. [Responses: a; b; c; d; e; f; g; h]

2. Is f in S or S'? [Responses: S; S']

3. Is there exactly one arrow leading from d? [Responses: Yes; No]

4. Is there exactly one arrow leading from each of a, b, c? [Responses: Yes; No]

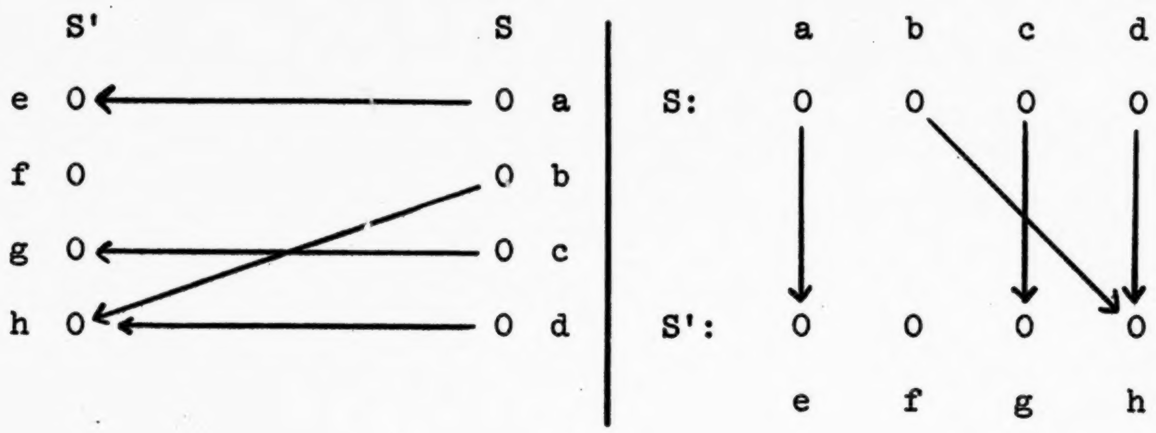
5. Do we have a transformation illustrated by the diagram? [Responses: Yes; No]

6. What is the domain of this transformation? [Responses: S; S'; {b, d}; {e, g, h}]

7. What element is in neither the domain nor the range of this transformation? [Responses: b; d; f; h]

8. Is this a transformation from S to S', from S' to S, or from {e, g, h} to S? [Responses: from S to S'; from S' to S; from {e, g, h} to S]

[Diagram below is to appear and remain through (8). Questions are Audio]



[Audio:] The scope shows the two previous transformations we have discussed. Let's look at them together for a moment. [Questions (1)-(8) following are audio:]

1. Both transformations have the same domain. What is this domain? [Responses: S; {e, f, g}; S']

2. Both transformations have the same range. What is this range? [*Responses*: *{e, g, h}*; S']

3. In both transformations, with what element of S' does a correspond? [*Responses*: *e*; f; g; h]

4. In both, with what element of S' does b correspond? [*Responses*: e; f; g; *h*]

5. With what element does c correspond? [*Responses*: e; f; *g*; h]

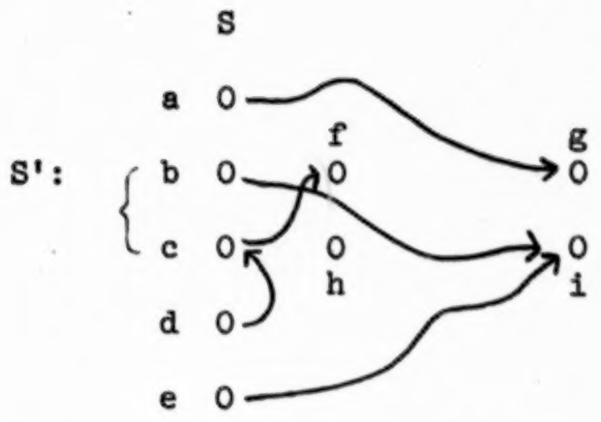
6. And d? [*Responses*: e; f; g; *h*]

7. How about e? [*Responses*: a; cannot be determined; *e is not in the domain of either transformation*]

8. So you see, then, both transformation are alike; they have the same *domain*; they have the same *range*; and in both a corresponds to e; b and d correspond to h; and c corresponds to g. Would it make any difference, therefore, *where* the sets S and S' are represented? [*Responses*: Yes, *No*]

[Diagram below appears and remains through (10). Questions (1)-(10) are to appear in written form below the diagram]

[*Audio*:] In all the examples of transformations considered so far, the sets S and S' have had no elements in common. But very often the sets S and S' do have common elements. In fact, in most of our work S and S' will be the *same* set. Let's look at the example shown on the scope.



[Audio:] Here S is the set {a, b, c, d, e}, and S' is the set {b, c, f, g, h, i}

1. What elements do S and S' have in common? [Responses: a; b; c; d; e; f; g; h; i]
2. Referring to the arrows as shown, what is the domain of the transformation indicated in the figure? [Responses: S; {b, c}; {a, b, e}; {c}; S']
3. What is the range of this transformation] [Responses: S'; {g, i}; {f, g, i}; {c}; {c, f, g, i}]
4. Are there any elements common to both the domain and range? [Responses: Yes; No]
5. What are these elements? [Responses: b; c; b and c; f; h]
6. What does an arrow leading from c tell you? [Responses: c is in the domain of the transformation; c is in the range of the transformation]
7. What does an arrow leading to c tell you? [Responses: c is in the domain of the transformation; c is in the range of the transformation]

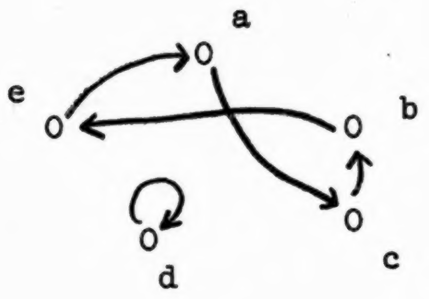
8. Does c lie in both the domain and the range of this transformation? [Responses: Yes; No]

9. Does h lie in either the domain or the range? [Responses: Yes; No]

10. How many members of S' in the range of this transformation do not appear in the domain? [Responses: 1; 3; 5; 7]

[Diagram below is to appear and remain through (10). Questions (1)-(10) are to appear in written form below the diagram]

[Audio:] In the next set of problems, we will not tell you what the sets S and S' are. You are to determine what they are using the arrows in the diagram shown, and watching where these arrows originate, and where they terminate.



1. The diagram illustrates a transformation from a set S to a set S'. What is the set S? [Responses: {a, b, c, e}; {a, b, c, d, e}; {d}; {e}]

2. Which of the elements of S does not have an arrow leading from it? [Responses: d; c and d; they all do]

3. What is the domain of this tranformation? [Responses: $\emptyset$, {a, b, c, e}; {d}; {a, b, c, d, e}]

4. How many elements are there in S'? [Responses: 1; 2; 3; 4; 5]

5. What is the range of this transformation? [*Responses*: {a, b, c, d}; {d}, *S*; none of these]

6. Are S and S' the same set here? [*Responses*: *Yes*; No]

7. Are the domain and range of this transformation the same set? [*Responses*: *Yes*; No]

8. Is there any element of S that corresponds to itself? [*Responses*: *Yes*; No]

9. What is this element? [*Responses*: a; b; c; *d*; e]

10. To what element does the element to which c corresponds, correspond? [*Responses*: a; b; c; d; *e*]

    [If *a*, *b*, *c*, *d*: *Audio*:] Let's break this down a bit.

    10-i) [*Audio*:] What is the element to which c corresponds? [*Responses*: a; *b*; c; e]

    10-ii) [*Audio*:] So now, to what element does b correspond? [*Responses*: a; b; c; d; *e*]

    10-iii) [*Audio*:] Therefore, to what element does the element to which c corresponds, correspond? [*Responses*: a; b; c; d; *e*]

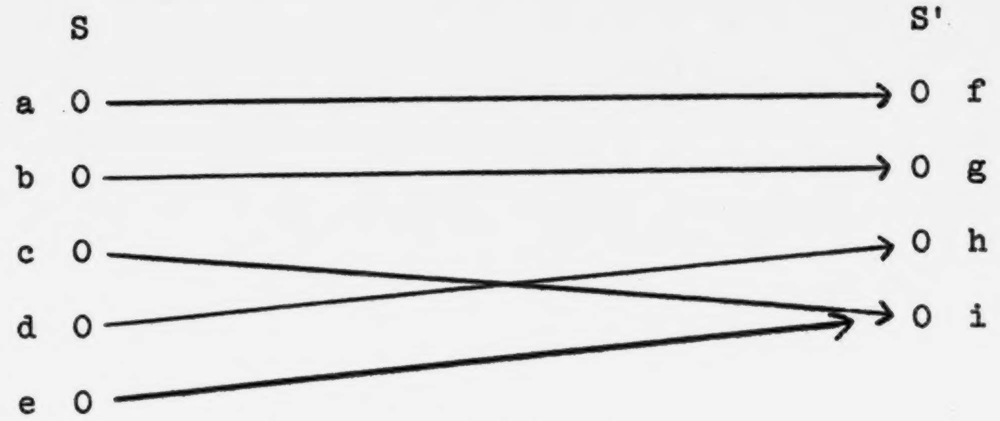
(Audio) Topic 3. Notation and Terminology

[Diagram below appears]

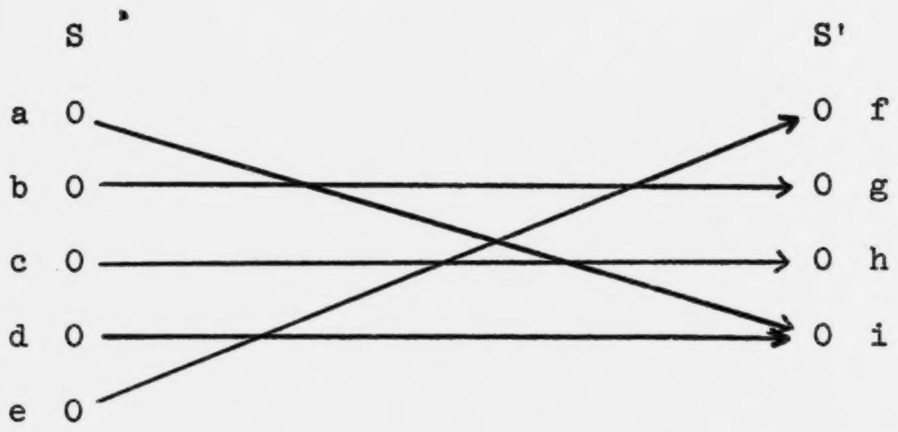
[Audio:] Let's consider the two sets S and S' shown on the scope

	S	S'	
a	O	O	f
b	O	O	g
c	O	O	h
d	O	O	i
e	O		

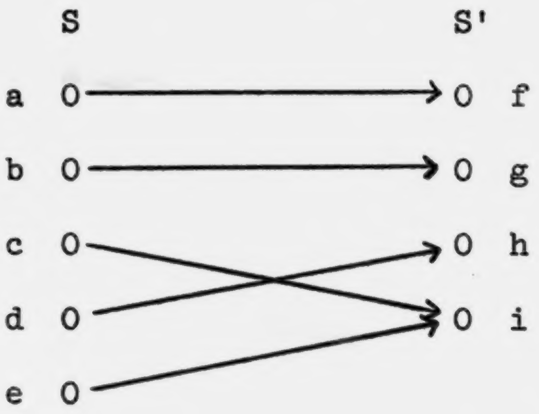
[Audio:] By our device using arrows, we can indicate one example of a transformation from S to S' as follows: [Here above diagram disappears and diagram below appears]

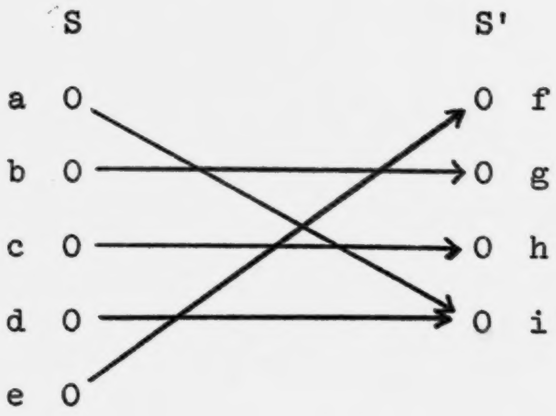


[Audio:] A second transformation from S to S' is shown next. [Here above diagram will vanish and the following diagram appears]

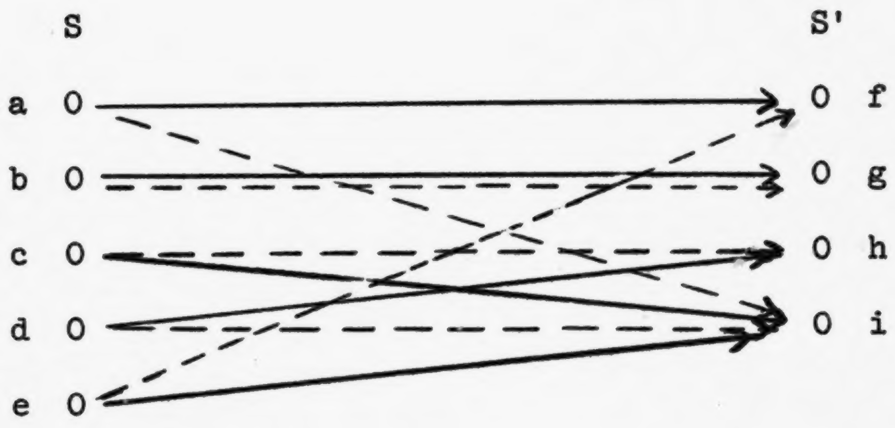


[Audio:] It should be clear that there are many transformations from S to S'. If we wanted to talk about the two transformations just shown, for example, we could show them one at a time. [Here clear the scope and show each one again, separately] Another way, of course, would be to show them side by side as you see on the scope:



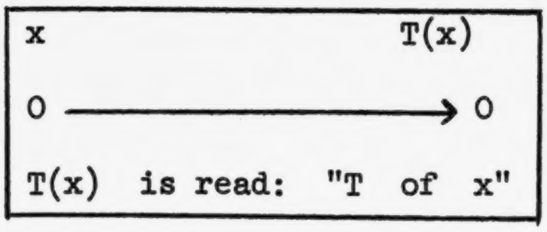


[Audio:] There are many situations in mathematics, however, where we want to talk at the same time about two or more transformations from a set S to a set S'. If we wanted to represent these on a single diagram, therefore, we would have to invent some device such as that shown in the following figure where one transformation is shown using solid arrows, the other using dotted arrows. [Here clear the scope and show the following figure]

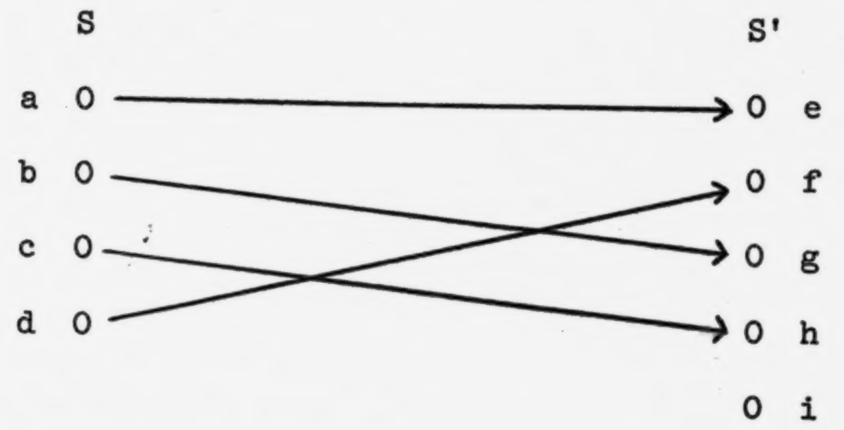


[*Audio*:] As you can see, diagrams of this sort can become very complicated and very hard to read--particularly if we had three or more transformations under study. Which elements in the set S are associated with which elements of S' would be hard to sort out.

[*Audio*:] For this reason, it turns out that we can make life a lot easier by giving the *transformation itself* a name. We do this by denoting a transformation by a capital letter, such as T, U, V, W, or by whatever symbol may be appropriate for the occasion. Furthermore, if T is a transformation from the set S to the set S', and if x is an element of S, then we denote the unique element of S' to which x corresponds under T by 'T(x)'. [Here clear the scope and show the following figure]

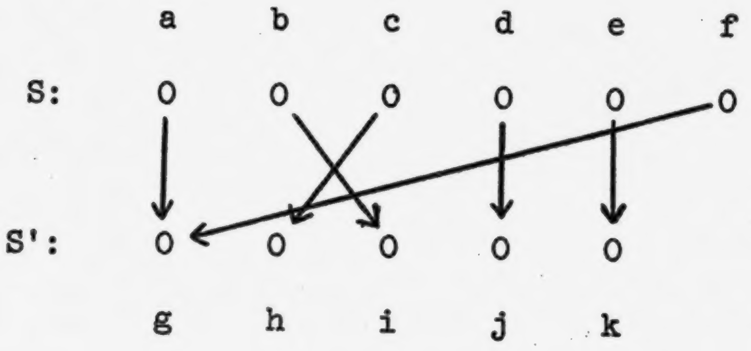


[*Audio*:] In the next diagram we show a transformation T from a set S to a set S'. [Here clear the scope and show the diagram below] At the right we have indicated with our notation how elements of S correspond to elements of S':



$T(a) = e$
$T(b) = g$
$T(c) = h$
$T(d) = f$

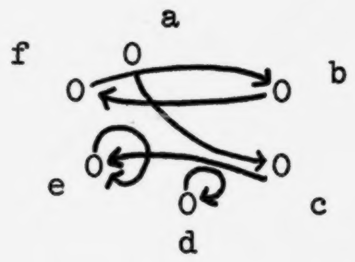
[Audio:] Now look at the following transformation shown on the scope. [Diagram below appears and remains through Question (7). Questions (1)-(7) will appear in written form below the diagram.]



1. Suppose we denote this transformation by T. What is T(a)? [Responses: g; h; i; j; k]
2. How would you write the element to which b corresponds? [Responses: b; h; T(b); T(i)]
3. Does $T(a) = T(f)$? [Responses: Yes; No]
4. What are $T(a)$ and $T(f)$ both equal to? [Responses: a; f; g; S']
5. What is $T(g)$? [Responses: a; f; a and f; undefined]
6. $x \in S$ and $y \in S$ and $x = y \Rightarrow T(x) =$ ___? [Responses: x; y; T(y); undefined]
7. If T is any transformation from any set S to any set S', and $x \in S$ and $y \in S$, does $x = y \Rightarrow T(x) = T(y)$? [Responses: Yes; No]

[Diagram below appears and remains through (18). Questions (1)-(18) will appear in written form under the diagram]

$S = S' = \{a, b, c, d, e, f\}$



1. Suppose we denote this transformation by W. What is W(c)?
   [*Responses*: a; b; c; d; *e*; f]
2. What is W(d)? [*Responses*: a; b; c; *d*; e; undefined]
3. To what element does the element to which a corresponds, correspond? [*Responses*: a; b; c; d; *e*]
4. What is W(W(a))? [*Responses*: b; c; d; *e*; f]
5. Is there any difference between W(W(a)) and the element which is the correspondent of the correspondent of a? [*Responses*: Yes; *No*]
6. Do you agree now that giving transformations names and denoting objects in the range by T(x) is easier? [*Responses*: *Yes*; No]
   [If *No*: *Audio*:] Don't be obstinate.
7. What is W(W(d))? [*Responses*: a; b; c; *d*; e; f]
8. What is W(W(f))? [*Responses*: a; b; c; d; e; *f*]
9. Does W(c) = W(e)? [*Responses*: *Yes*; No]
10. Is c = e? [*Responses*: Yes; *No*]
11. If x and y are in S, does W(x) = W(y) imply x = y?
    [*Responses*: Yes; *No*]

12. If x and y are in S, does $x = y$ imply $W(x) = W(y)$?

[Responses: Yes; No]

13. If x and y are in S and $x = y$, why does $W(x) = W(y)$?

[Responses: Because $a = a$ and $b = b$; because W is a transformation; because the domain of W is S]

14. What is $W(W(W(a)))$? [Responses: a; b; c; e; f]

15. Is $f = W(b)$? [Responses: Yes; No]

16. Is $b = W(f)$? [Responses: Yes; No]

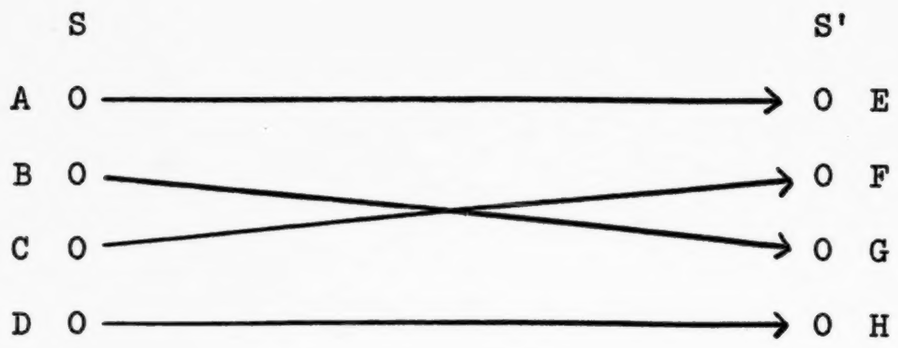
17. Could we say that W "interchanges" b and f? [Responses: Yes; No]

18. Is there any other pair of elements of S that W "interchanges"?

[Responses: Yes; No]

[Diagram below appears and remains through (7). (1)-(7) appear in written form under the diagram]

[Audio:] In addition to giving names to transformations, it also is very useful to simplify matters by referring to $T(x)$ as the "image of x under the transformation T." Look at the diagram shown on the scope:



[Audio:] Here, if we denote this transformation by T, then G, for example, would be $T(B)$. We could also say that "G is the image of B under T." Let's see some quick answers to the following questions now.

1. What is T(C)? [*Responses*: A; E; *F*; G; H]
2. What is the image of C under T? [*Responses*: E; *F*; G; H]
3. Does T(C) = image of C under T? [*Responses*: *Yes*; No]
4. What is the image of D under T? [*Responses*: E; F; G; *H*]
5. What is the image of F under T? [*Responses*: A; B; C; D; *undefined*]
6. What shorthand symbolic form would we use for: "The image of A under T is E"? [*Responses*: ATE; *T(A) = E*; T(E) = A; A(T) = E]
7. What shorthand symbolic form would we use for: "The image of B under T is G"? [*Responses:* *G = T(B)*; T(G) = B; B(T) = G; (T)B = G]

   [Diagram below appears and remains through (9). Questions (1)-(9) appear in written form under the diagram]

[*Audio*:] Let's consider one more problem dealing with our new notation and terminology. Suppose S is the set of integers *one* through *five*, and S' is the set of integers *one* through *seventy-nine*. Suppose U is a transformation from S into S' as shown in the diagram:

S: 1, 2, 3, 4, 5

S': 1, 2, 3, ... , 77, 78, 79

$$U(x) = 3x^2 + 1$$

Answer the following questions.

1. What is U(2)? [*Responses*: 8; *13*; 37]
2. What is U(5)? [*Responses*: 75; *76*; 225; 226]

3. What is the image of 4 under U? [Responses: U(1); U(U(1)); 25; 145]

4. What is U(U(1))? [Responses: U(1); the image of 4 under U; 25; 145]

5. What is the image of 6 under U? [Responses: 109; 325; undefined]

6. How many elements are there in the range of U? [Responses: 1; 3; 5; 7; 79

7. How many elements are there in the domain of U? [Responses: 1; 3; 5; 7; 79]

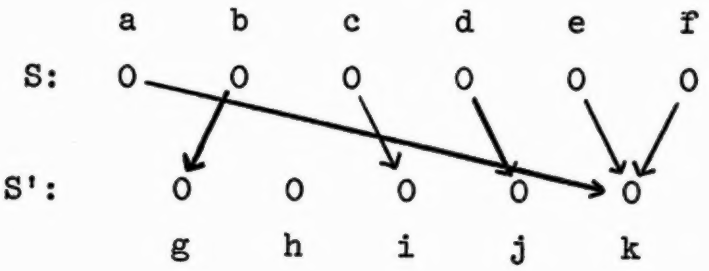
8. How many elements in S'? [Responses: 1; 3; 5; 7; 79]

9. Is S' = range of U? [Responses: Yes; No]

LESSON 2

[Audio] Topic 0. Review

[Audio:] Let's review a few of the concepts we discussed in the last lesson. Study the diagram that appears on the scope.



[Questions (1)-(12) are Audio]

1. Which elements in the domain of this transformation correspond to more than one element in the range? [Responses: None; h; k; a and e and f]
2. By the definition of a transformation from S to S', each element in S must correspond to how many elements of S'? [Responses: At least one; exactly one; sometimes two]
3. In the transformation above, what does d correspond to? [Responses: g; h; i; j; h]
4. What is the domain of this transformation? [Responses: S; S'; {b, c, d}; {g, i, j}; {g, i, j, k}]
5. List the elements in the domain. [Responses: a; b; c; d; e; f; g; h; i; j; k]
6. Is there an arrow leading to k? [Responses: Yes; No]
7. Is k in the domain of the transformation? [Responses: Yes; No]
8. Is k in the range of this transformation? [Responses: Yes; No]

9. Is h in the range? [Responses: Yes; [✓ No]]

10. What is the range of this transformation? [Responses: S; S'; {b, c, d}; {g, i, j}; [✓ {g, i, j, k}]]

11. How many elements of S correspond to the same element k in the range? [Responses: 0; 1; 2; [✓ 3]; 6]

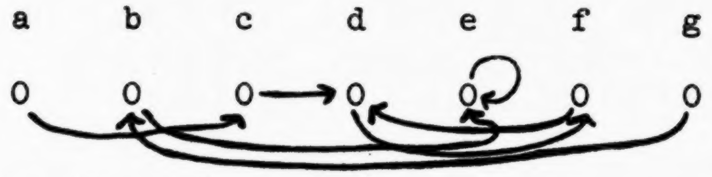
12. List these elements. [Responses: [✓ a]; b; c; d; [✓ e]; [✓ f]]

[Diagram below appears, and is to remain through (18).]

[Audio:] Look at this transformation. Let's call it T.

$$S = S' = \{a, b, c, d, e, f, g\}$$

(T):



[Questions (1)-(18) will appear in written form on the screen below the above diagram.]

1. What is $T(b)$? [Responses: c; d; [✓ e]; f; g]

2. What is $T(c)$? [Responses: a; c; [✓ d]; e; f]

3. How many elements are there in the domain of T? [Responses: 3; 4; 5; 6; [✓ 7]]

4. List the elements that are not in the range of T. [Responses: [✓ a]; b; c; d; e; f; [✓ g]]

5. What is the range of T? [Responses: S; S'; {c}; {a, g}; [✓ {b, c, d, e, f}]]

6. What is the image of e under T? [Responses: b, [✓ e]; $T(e)$ is undefined].

7. Is there an $x \in S$ such that $x \neq e$ and $T(x) = x$? [Responses: Yes; [✓ No]]

8. What is Domain of $T \cap$ Range of T? [*Responses*: S; *Range of T*; S'; $\{a, g\}$]

9. Is there an $x \in S$ such that $T(T(x)) = x$? [*Responses*: *Yes*; No]

10. List the elements x of S such that $T(T(x)) = x$. [*Responses*: a; b; c; *d*; *e*; *f*; g]

11. What is the image under T of the image under T of d? [*Responses*: b; c; *d*; e; f; g]

12. What is $T(T(d))$? [*Responses*: b; c; *d*; e; f; g]

13. What is $T(T(f))$? [*Responses*: b; c; d; e; *f*; g]

14. If $x \in S$ and $T(T(x)) = T(x)$, what is $T(x)$? [*Reponses*: a; c; *e*; g; no such x]

15. What is $T(T(T(T(T(a)))))$? [*Responses*: a; b; c; d; e; *f*; g; h]

16. Suppose T is any tranformation from some set S to a set S'. Does $T(x) = T(y) \Rightarrow x = y$? [*Responses*: Yes, by the definition of a transformation; *No, not necessarily*]

17. Suppose T is any transformation from some set S to a set S'. Does $x = y \Rightarrow T(x) = T(y)$? [*Responses*: *Yes, by the definition of a transformation*; No, not necessarily]

18. If T is a transformation from a set S to a set S', what is the range of T? [*Responses*: S; S'; *$\{T(x) \mid x \in S\}$*; none of these]

[Diagram in ☐ below is to appear and remain through (10).]

[*Audio*:] Here's an example of a transformation of a somewhat different sort. Let's call it E.

$S = \{x \mid x$ was a president of U.S. before L.B.J.$\}$

$S' = \{0, 1, 2, 3, 4, 5\}$

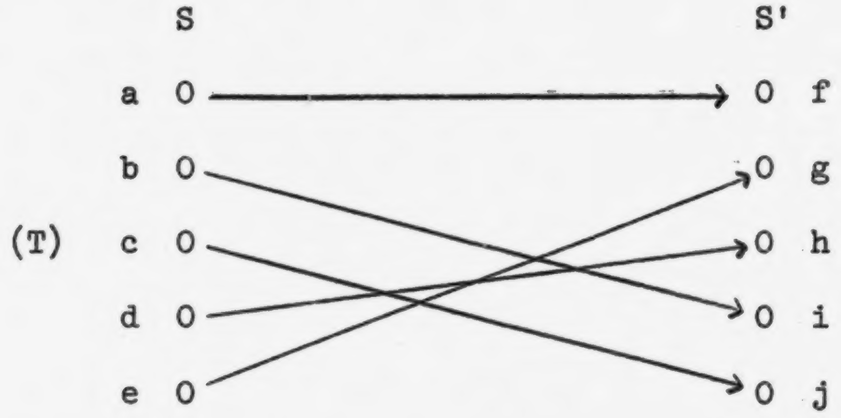
$E(x)$ = maximum number of times x was elected

[Questions (1)-(10) appear in written form on the screen below the above diagram.]

1. What is E(Eisenhower)? [*Responses*: 0; 1; *2*; 3]
2. What is the image of Kennedy under E? [*Responses*: 0; *1*; 2; 3; 4]
3. What is E(A. Johnson)? [*Responses*: *0*; 1; 2; 3]
4. What is the range of E? [*Responses*: S'; {0, 1, 2, 3, 4}; {1, 2, 3, 4}; *{0, 1, 2, 4}*; {1, 2, 4}]
5. If $E(x) = 4$, what is x? [*Responses*: Washington; Lincoln; Cleveland; T. Roosevelt; *F. Roosevelt*]
6. What is E(L. Johnson)? [*Responses*: 0; 2; *L. Johnson $\notin$ S*; We don't know yet]
7. What is E(4)? [*Responses*: *Undefined*; F. Roosevelt; a four-year term]
8. What is the image under E of Truman? [*Responses*: 0; *1*; 2; 3]
9. If $x = y$, does $E(x) = E(y)$? [*Responses*: *Yes*; No]
10. If $x \neq y$, does $E(x) \neq E(y)$? [*Responses*: Yes; *No*]

[*Audio*] Topic 1. *Equality of two transformations*

[*Audio*:] In all of our lessons we will be working constantly with transformations. We will be defining various transformations, we will be studying their properties, we will be learning how to apply them in a number of different geometric situations, and we will be learning how to generate new transformations out of old ones. The emphasis in all this will be to think of a transformation as a *type of mathematical object*--just as numbers, triangles, planes, etc., are mathematical objects. This being so, we want to be able to say when two or these objects, or transformations, are *equal*. So, let's see if we can come to an agreement on when we would call two transformations equal. Look at the transformation T shown on the scope:

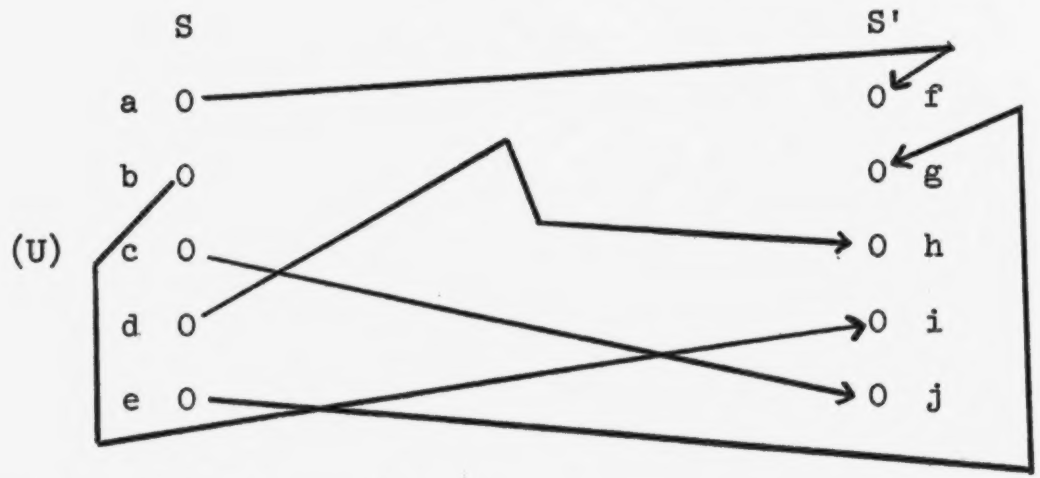


You can see from the diagram that T has domain S, range S', and $T(a) = f$, $T(b) = i$, and so on. We can summarize all relevant information about T in a box, as shown now in this diagram:

Data about T

Domain T = S	T(a) = f
Range T = S'	T(b) = i
	T(c) = j
	T(d) = h
	T(e) = j

[*Audio*:] Now let's look at *this* transformation, which we will call U:



[Questions (1)-(4) are *Audio*]

1. What is the domain of U? [*Responses*: {c, g}; {a, b, d}; *S*; S']
2. What is the range of U? [*Responses*: {c, e}; S; {i, j}; {f, h, i, j}; *S'*]
3. What is U(e)? [*Responses*: f; *g*; h; i]
4. What is the image of f under U? [*Responses*: a; b; i; j; *undefined*]

[*Audio*:] As we did earlier, we can summarize all the relevant information about U in a box, as shown in this diagram:

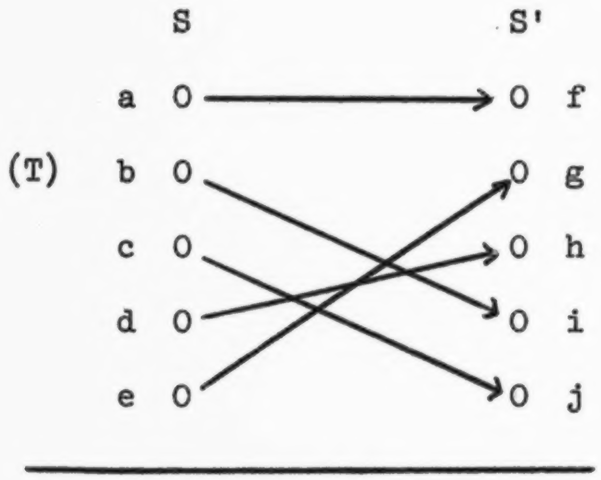
Data about U

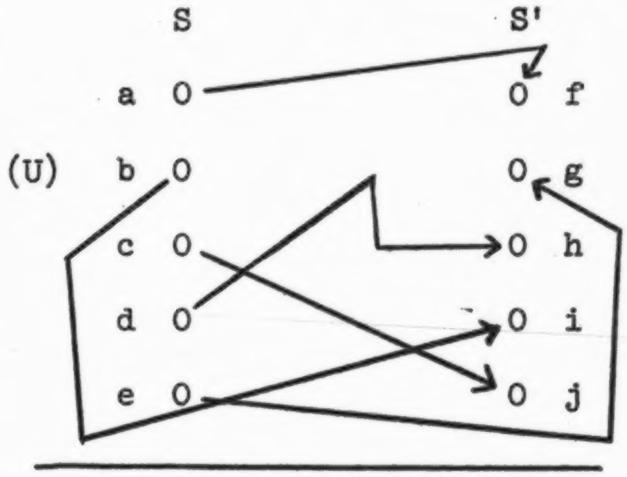
Domain U = S
Range U = S'

U(a) = f
U(b) = i
U(c) = j
U(d) = h
U(e) = g

[Diagram above disappears, and tableau below is to appear]

[Audio:] Let's look now at our two transformations T and U side by side along with the data about them found in the boxes on the screen.





Data about T

Domain T = S | T(a) = f
| T(b) = i
| T(c) = j
Range T = S' | T(d) = h
| T(e) = g

Data about U

Domain U = S | U(a) = f
| U(b) = i
| U(c) = j
Range U = S' | U(d) = h
| U(e) = g

[Questions (1)-(8) will appear in written form on the screen below the diagrams.]

1. Do T and U have the same domain? [Responses: Yes; No]
2. Do T and U have the same range? [Responses: Yes; No]
3. Are T and U both transformations from S to S'? [Responses: Yes; No]
4. Does T(a) = U(a)? [Responses: Yes; No]
5. Does b have the same image under both T and U? [Responses: Yes; No]
6. Does T(c) = U(c)? [Responses: Yes; No]
7. Do all of c, d, and e have the same images under both T and U? [Responses: Yes; only c]
8. Does any element of S correspond to different elements under T and U? [Responses: Yes; No]

[Audio:] We see, therefore, that both T and U have the same domain, the same range, and--most important--every element in this common domain has the same image in the range under both T and U. But if we look at our figures illustrating T and U, we might feel that somehow T and U are not the "same." However, let's look at an analogy from arithmetic: [Digrams above disappear; Questions (1)-(3) below are to appear in written form on the screen]

1. What is the smallest even positive integer? [Responses: 0; $\frac{1}{2}$; 1; 2; 4]
2. What is the first non-square positive integer? [Responses: 1; 2; 3; 4; 5]
3. What is the only positive integer that is an even prime? [Responses: 2; 4; 6; 9; 16]

[Diagram below appears now, and remains through the following paragraph]

[*Audio*:] Look at the scope. Notice that all three descriptions apply to the number 2, and only to the number 2:

The number 2 is:

(a) the smallest even positive integer;

(b) the first non-square positive integer;

(c) the only positive integer that is an even prime.

So, even though these are three different descriptions, they all refer to the number 2. That is, 2 can be described in any one of these three different ways. Let's take this clue from arithmetic, therefore, and agree to *disregard* the way in which the arrows in our diagrams on transformations are drawn. This means that we will say two transformations T and U from a set S to a set S' are *equal* if T and U have the same domain, the same range, and $T(x) = U(x)$ for all elements x of S. The various parts of this definition are summarized in the figure on the scope: [Here above diagram is replaced by the diagram below]

T *and* U *are transformations from* S *to* S'

$T = U$ if and only if

1) Domain T = Domain U

2) Range T = Range U

3) $T(x) = U(x)$ for all $x \in S$.

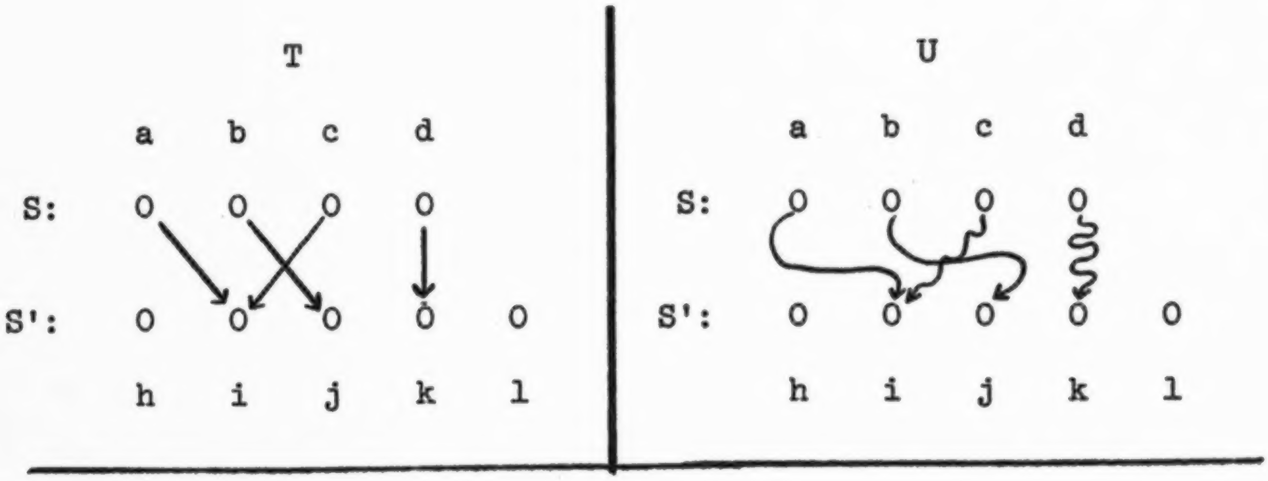
Let's look at our two transformations shown earlier. [Here the side-by-side diagrams of T and U are to be shown again, without the data charts accompanying each. They are to remain through (6)]

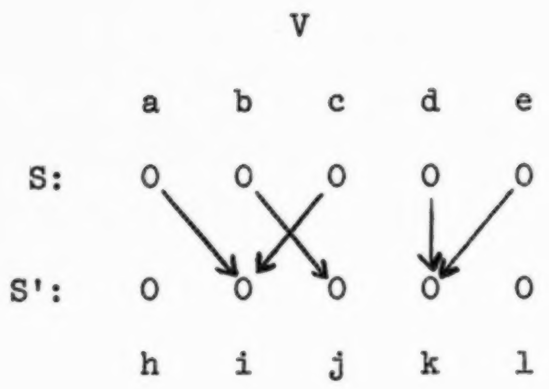
[Questions (1)-(6) are Audio]

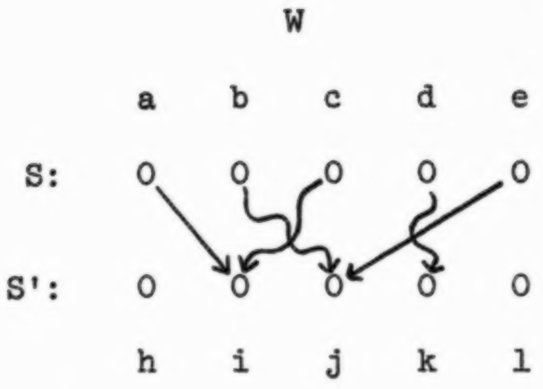
1. Once again, do T and U have the same domain? [Responses: Yes; No]
2. Do they have the same range? [Responses: Yes; No]
3. Does $T(x) = U(x)$ for all x in S? [Responses: Yes; No]
4. Does the 'straightness' or 'waviness' of the arrows affect the equality or inequality of T and U? [Responses: Yes; No]
5. In determining whether $T = U$, are we interested in any intermediate effects, or only in knowing whether $T(x) = U(x)$ for all $x \in S$? [Responses: The intermediate effects are important; We are concerned only with whether $T(x) = U(x)$ for all $x \in S$]
6. Does $T = U$? [Responses: Yes; No]

[Audio:] Let's try a few problems.

[Diagram above disappears and diagram below appears and is to remain through Question 48]







[Questions (1)-(17) appear in written form below these diagrams]

1. Look at transformation T. What is T(a)? [Responses: i; j; k]
2. Which two elements of S have the same image under T? [Responses: [a, i]; [b, c]; [a, c]; [h, l]]
3. The element k of S' is the image under T of which elements of S? [Responses: a; a and c; b and d; d]
4. What is the domain of T? [Responses: {a, b, c, d}; {i, j, k}; {h, l}; S']
5. What is the range of T? [Responses: {a, b, c, d}; {i, j, k}; {h, l}; S']
6. Look at transformation V. What is V(a)? [Responses: i; j; k]
7. What other element of S has the same image as a under V? [Responses: b; c; d]
8. The element k of S' is the image under V of which elements of S? [Responses: a; a and c; d; e; d and e]
9. What is the range of V? [Responses: {a, b, c, d, e}; {i, j, k}]
10. Does T(a) = V(a)? [Responses: Yes; No]
11. Does T(b) = V(b)? [Responses: Yes; No]
12. Does c have the same image under T and V? [Responses: Yes; No]

13. Does $T(d) = V(d)$? [Responses: Yes; No]

14. Do T and V have the same range? [Responses: Yes; No]

15. Does $T(x) = V(x)$ for all $x \in$ domain T? [Responses: Yes; No]

16. By our definition of equality of two transformations, does $T = V$? [Responses: Yes; No]

[If YES: Audio:] Did we fool you on this one? Think again: What must be true if two transformations are to be equal? Try it again.

17. What is the domain of V? [Responses: $\{a, b, c, d\}$; $\{a, b, c, d, e\}$; $\{i, j, k\}$]

18. So, do T and V have the same domain? [Responses: Yes; No]

19. And if two transformations are equal, must they have the same domain? [Responses: Yes; No]

20. Once again: Does $T = V$? [Responses: Yes; No]

21. Why does $T \neq V$? [Responses: Because T and V have different domains; because T and V have different ranges; because there is an x in $\{a, b, c, d\}$ such that $T(x) \neq V(x)$]

22. O.K., we know $T \neq V$. Are there any two of these four transformations that are equal? [Responses: Yes; No]

[If NO: Audio:] Better look again! Try the question once more.

23. Is $U = V$? [Responses: Yes; No]

24. Is $T = W$? [Responses: Yes; No]

25. Is $V = W$? [Responses: Yes; No]

26. Is $V = T$? [Responses: Yes; No]

27. Is $T = U$? [Responses: Yes; No]

28. Do T and U have the same domain? [Responses: Yes; No]

29. How many elements in the range of T are not in the range of U? [Responses: 0; 1; 2; 3; 5]

30. Do T and U have the same range? [Responses: Yes; No]

31. Does $T(c) = U(a)$? [Responses: Yes; No]

32. Is "$T(c) \neq U(a)$" true or false? [Responses: True; false]

33. Is there an $x \in \{a, b, c, d\}$ with $T(x) \neq U(x)$? [Responses: Yes, No]

34. How many elements x of $\{a, b, c, d\}$ satisfy '$T(x) = U(x)$'? [Responses: Exactly one; at most three; exactly three; at least four]

35. Once again, is $T = U$? [Responses: Yes; No]

36. Do the intermediate effects--represented by the 'straightness' or 'waviness' of the arrows--play any role in determining whether $T = U$? [Responses: Yes; No]

    [If YES: Audio:] Better think again about the definition of equality of two transformations. Let's try it once more.

37. You said a minute ago that $V \neq W$. But do V and W have the same domain? [Responses: Yes; No]

38. List the elements in the common domain of V and W. [Responses: a; b; c; d; e; f; g; h; i; j; k; l]

39. And do V and W have the same range? [Responses: Yes; No]

40. List the elements in the common range of V and W. [Responses: a; b; c; d; e; f; g; h; i; j; k; l]

41. V and W have the same domain and the same range. Do you still maintain that $V \neq W$? [Responses: Yes; No]

[If NO: Audio:] Well, V and W do have the same domain and the same range. Is this enough to be sure that $V = W$? Think a little bit more here, and try the question again.

42. Does $V(x) = W(x)$ for all x in the common domain $S = \{a, b, c, d, e\}$? [Responses: Yes; No]

43. List all elements x in S such that $V(x) \neq W(x)$. [Responses: a; b; c; d; e]

44. What is $V(e)$? [Responses: h; i; j; k; l]

45. What is $W(e)$? [Responses: h; i; j; k; l]

46. Do all other elements of the common domain have the same image under both V and W? [Responses: Yes; No]

47. But does $V(e) = W(e)$? [Responses: Yes; No]

48. [Audio:] There's a moral here; two transformations M and N can have the same domain, the same range, and still not be equal. Why? [Responses: Because there may be straight arrows in one case, wavy in the other; because the intermediate effects may be different; because there may be at least one element x in the domain of M and N with $M(x) \neq N(x)$; because $M(x)$ may not equal $N(x)$ for any x in the domain of M and N]

[Diagram on page 10 is to disappear, and diagram below is to appear and remain through (4) below]

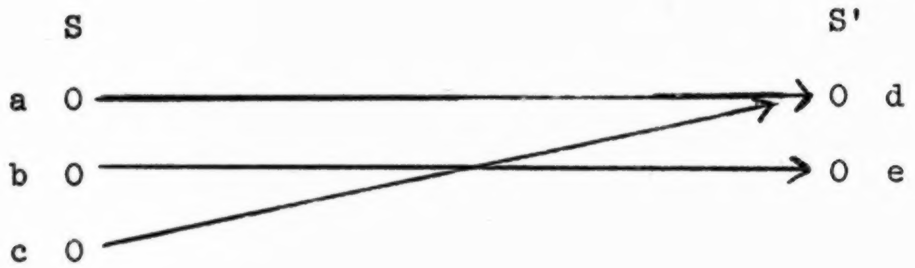
[Audio:] We saw that two transformations may have the same domain and the same range, and still not be equal. This will happen if there is at least one element in the domain having different images under the two transformations. We can use this observation to determine how many distinct (that is, unequal) transformations there are from a set S having three elements to a set S' having two elements. Look at the figure shown on the scope:

S		S'
a O		O d
b O		O e
c O		

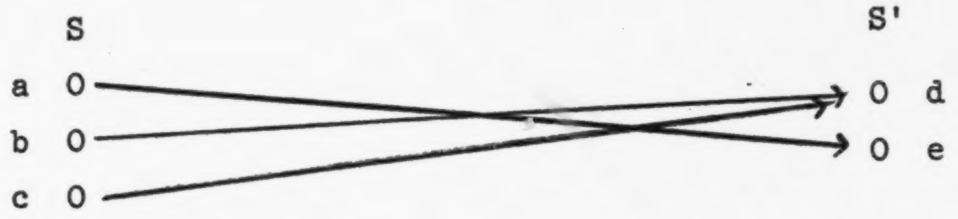
[Questions (1)-(4) will appear in written form on the screen below the above diagram]

1. Suppose we wanted to construct a transformation T from S to S'. How many choices would there be for T(a)? [Responses: 1; 2; 3; 5]
2. List the choices we could make for T(a). [Responses: a; b; c; d; e]
3. How many choices are there for T(b)? [Responses: 0; 2; 3; 5]
4. How many choices for T(c)? [Responses: 0; 1; 2; 3]

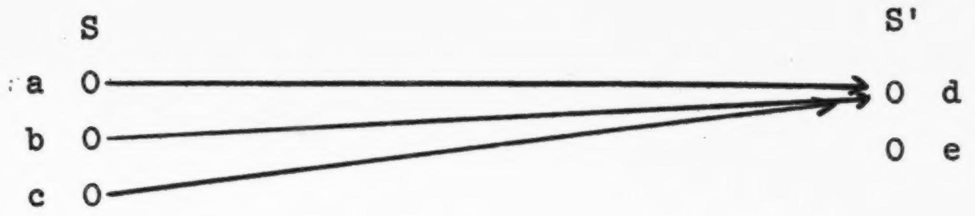
[Audio:] Let's suppose we chose T(a) = d, T(b) = e, and T(c) = d. Then our transformation would appear as shown on the scope: [After audio, diagram below now appears to the right of the "unarrowed" diagram which is to remain]



On the other hand, if we chose T(a) = e, T(b) = d and T(c) = d, then our transformation would appear as in the following: [after audio, diagram below appears at lower left; the "unarrowed" diagram and the first transformation remain]



[Audio:] We could get a third transformation from S to S' by having d be the image of all the elements a, b, and c: [After audio, diagram below appears at lower right; previous diagrams remain]



[Questions (1)-(7) will appear in written form on the screen underneath the above diagram]

1. Look at the first two transformations ((1) and (2)) shown on the scope. Are they equal? [Responses: Yes; No]
2. Is there at least one element in S which has different images under these transformations? [Responses: Yes; No]
3. What is this element? [Responses: a; b; c; d; e]
4. Do these two distinct transformations correspond to two distinct ways of choosing T(a), T(b) and T(c) jointly? [Responses: Yes; No]
5. Look at the second and third transformations ((2) and (3)) shown on the scope. Are they equal? [Responses: Yes; No]
6. Which element in S has different images under these two transformations? [Responses: a; b; c; d; e]
7. Do these two distinct transformations correspond to two distinct ways of choosing T(a), T(b), and T(c) jointly? [Responses: Yes; No]

[Diagram above is to disappear now. Questions (1)-(6) will appear in written form on the screen. The earlier "unarrowed" diagram is to reappear centered on the screen, and remain through (6)]

1. Let's try to generalize these ideas a bit. Look at the diagram. How many ways are there of choosing each of T(a), T(b), and T(c) separately? [Responses: 2; 3; 5]
2. Therefore, how many distinct ways are there of choosing T(a), T(b) and T(c) jointly? [Responses: 2; $2 \cdot 2$; $2 + 2 + 2$; $2 \cdot 2 \cdot 2$]
3. Therefore, how many distinct transformations are there from S to S'? [Responses: 2; $2 \cdot 2$; $2 + 2 + 2$; $2 \cdot 2 \cdot 2$]
4. How many of the eight transformations T from S into S' satisfy: T(a) = d? [Responses: 0; 2; 4; 6]
5. How many of the eight transformations T from S to S' satisfy T(a) = T(b)? [Responses: 0; 1; 4; 5; 8]
6. How many of the eight transformations T from S to S' satisfy T(a) = T(b) = T(c)? [Responses: 1; 2; 3; 5; 8]

[Diagram above disappears now. Questions (1)-(9) will appear in written form on the screen]

1. Suppose $S = \{a, b, c\}$ and $S' = \{d, e, f, g\}$. If T is a transformation from S to S', how many choices are there for T(a)? [Responses: 0; 2; 4; 8; 16]
2. How many choices for each of b and c? [Responses: 0; 2; 4; 8; 16]

3. How many distinct ways are there of choosing $T(a)$, $T(b)$, and $T(c)$ jointly? [Responses: $3 + 4$; $3 \cdot 4$; $\underline{4^3}$; 3^4]

4. How many distinct transformations are there from a set having three elements to a set having four elements? [Responses: 7; 12; $\underline{64}$; 81]

5. Suppose S has three elements and S' has n elements. How many distinct transformations are there from S to S'? [Responses: $3 + n$; $3n$; $\underline{n^3}$; 3^n]

6. Suppose S has m elements and S' has three elements. How many distinct transformations are there from S to S'? [Responses: $3 + m$; $3m$; m^3; $\underline{3^m}$]

7. Finally, suppose S has m elements and S' has n elements. How many distinct transformations are there from S to S'? [Responses: $n + m$; $n \cdot m$; m^n; $\underline{n^m}$]

8. Suppose S has m elements and S' has n elements. If $a \in S$ and $a' \in' S'$, how many distinct transformations T are there such that $T(a) = a'$? [Responses: m^n; n^m; $m^n - 1$; $(m-1)^{n-1}$; $\underline{(n-1)^{m-1}}$]

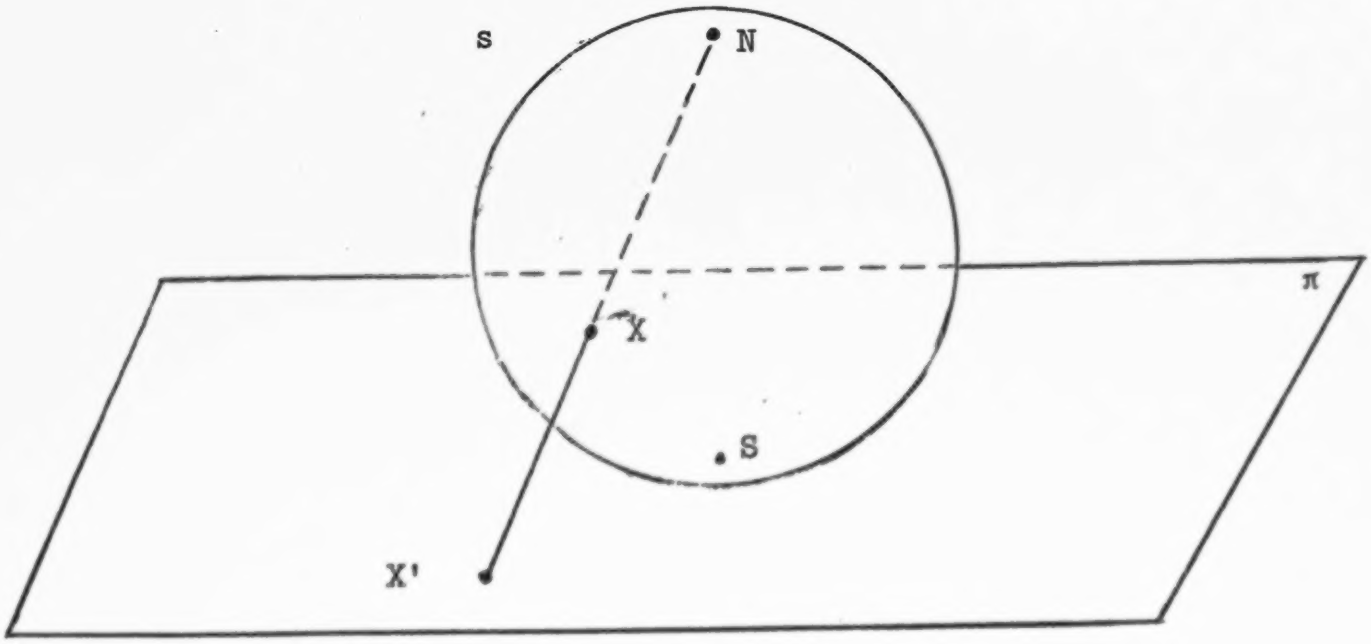
9. Suppose S has m elements and S' has n elements. If a and b are in S and a' is in S', how many distinct transformations T are there such that $T(a) = T(b) = a'$? [Responses: $n^m - 2$; $m^n - 2$; $(m-2)^{n-1}$; $\underline{(n-1)^{m-2}}$; $(n-2)^{m-1}$; $(m-1)^{n-2}$]

[Audio] Topic 2. Some further examples: The existence of transformations and the problem of definition

[Diagram below appears after above audio]

[Audio:] I think maybe we need a change of pace. Let's get away from our "dots-and-arrows" transformations and look at a few transformations from geometry. The transformation shown on the scope is called stereographic projection. As the picture indicates, lower case s is a sphere; this sphere is tangent to a plane π at a point capital S we may call the "south pole" of the sphere. The point N which is antipodal to S we may call the "north pole" of the sphere.

St: Stereographic projection



[Audio:] Now, let us take a point X on the sphere different from N and construct the line NX. Let us call the point where the line NX intersects the plane π the point X'. Then the transformation which associates X to X' is called stereographic projection; we will denote this transformation by (capital) St.

[Questions (1)-(25) will appear in written form on the screen; the above diagram will remain through (25)]

1. What is the domain of St? [*Responses*: s; *$s - \{N\}$*; π; $\pi - \{S\}$]
2. Is there a unique line joining N and N? [*Responses*: Yes; *No*]
3. If $X = N$, could we determine X'? [*Responses*: Yes; *No*]
4. What is St (N)? [*Responses*: N; S; *N is not in the domain of St*]

[*Audio*:] Do you remember that a while back we had an example of a transformation from a *proper* *subset* of a certain set to another set? Well, here is an example like that: Stereographic projection is a transformation from the proper subset $s - \{N\}$ of the sphere s to the plane π.

5. Is $St(X') = X$? [*Responses*: Yes; *No*]
6. Is $St(X) = X'$? [*Responses*: *Yes*; No]
7. What is St(S)? [*Responses*: *S*; N; S is not in the domain of St]
8. Is S in π? [*Responses*: *Yes*; No]
9. Is S on the line NS? [*Responses*: *Yes*; No]
10. Is $S \in NS \cap \pi$? [*Responses*: *Yes*; No]
11. Is $S = St(S)$, then? [*Responses*: *Yes*; No]
12. Suppose $St(X) = X$. What can you say about X? [*Responses*: $X = N$; there is no such point X; *X is on both the sphere and the plane*]
13. What is the only point on both the sphere and the plane? [*Responses*: N; *S*; there is no such point]

14. Therefore, if $St(X) = X$, what can you say about X? [Responses: $X = N$; there is no such point X; $X = S$]

15. Is S the only point whose image under St is itself? [Responses: Yes; No]

16. What is the range of St? [Responses: X'; S; $\{X', S\}$; a subset of π]

17. Take a point $Z \in \pi$. What would happen if $NZ \cap s = \{N\}$? [Responses: $N = S$; NZ would be tangent to s at N; NZ would be tangent to s at Z]

18. If NZ is tangent to s at N, does NZ lie in the plane π' tangent to s at N? [Responses: Yes; No]

19. Is $\pi' \parallel \pi$? [Responses: Yes; No]

20. But if $Z \in \pi$ and $NZ \subset \pi'$, is $Z \in \pi \cap \pi'$? [Responses: Yes; No]

21. Can we have both $\pi' \parallel \pi$ and $Z \in \pi \cap \pi'$? [Responses: Yes; No]

22. Therefore, if $Z \in \pi$, does $NZ \cap s = \{N\}$? [Responses: Yes; No]

23. If $Z \in \pi$, must there be a second point $X \neq N$ such that $X \in NZ \cap s$? [Responses: Yes; No]

24. What is $St(X)$? [Responses: N; S; X'; Z]

25. What have we proved about the range of St? [Responses: It is parallel to π; it is a proper subset of π; it is equal to π]

[Audio:] O.K., so far you'r right with it, Man! But I think we should go back to the beginning and check that we actually have a transformation. We cannot talk, for example, about the range of the transformation St unless we first know St is a transformation. The next series of questions will help us check on this point.

[Questions (1)-(19) will appear in written form on the screen. Diagram on page 19 is to remain through (19)]

1. Given X on s and different from N, what was X'? [Responses: Midpoint of segment NX; intersection of the plane NXS with π; intersection of the line NX with π]
2. What would happen if $NX \cap \pi = \emptyset$? [Responses: $N \notin s$; $NX \parallel \pi$; N would not be in the domain of St]
3. If $NX \parallel \pi$, would NX lie in the plane π' tangent to s at N? [Responses: Yes; No]
4. But if $NX \subset \pi'$, is $X \in \pi'$? [Responses: Yes; No]
5. Is $X \in s$? [Responses: Yes; No]
6. If $X \in \pi'$ and $X \in s$, is $X \in s \cap \pi'$? [Responses: Yes; No]
7. However, what is $s \cap \pi$? [Responses: $\{S\}$, $\{N\}$, $\{S, N\}$; $\emptyset$]
8. Therefore, if $X \in s \cap \pi'$, what could we say about X? [Responses: $X = S$; $X = N$; X is undefined]
9. Does $X = N$ by our choice of X? [Responses: Yes; No]
10. Therefore, if $X \in s - \{N\}$, is $NX \parallel \pi$? [Responses: Yes; No]
11. Is $NX \cap \pi = \emptyset$? [Responses: Yes; No]

[Audio:] We are now clear about one thing: If X is in $s - \{N\}$, then the line NX intersects the plane in at least one point. But remember, to have a transformation, we have to check still that NX cannot intersect π in more than one point.

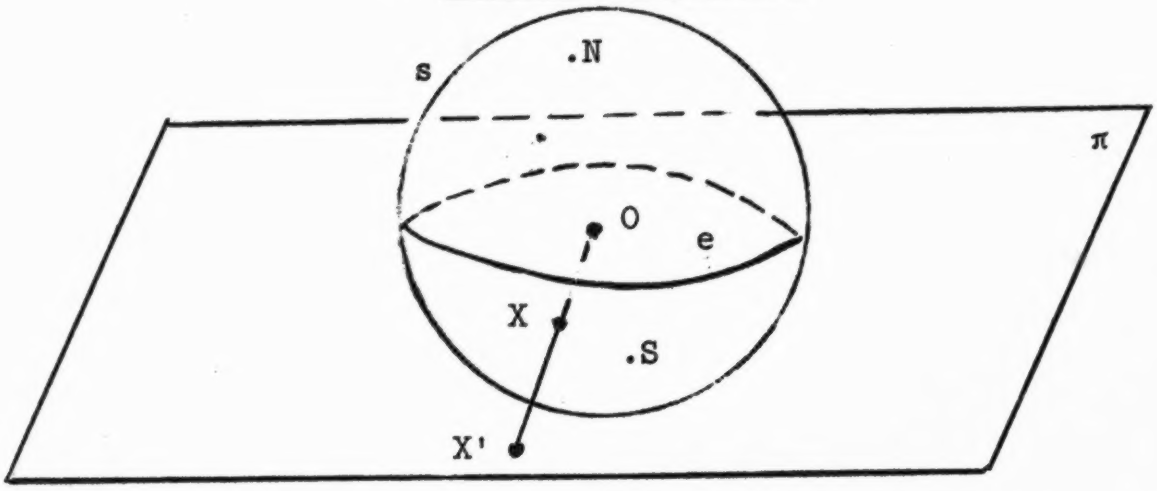
12. Suppose NX intersects π in two distinct points X' and X''. Will X' and X'' both be in π? [Responses: Yes; No]

13. And if two points X' and X'' lie in a plane π, what can we say about the line $X'X''$ containing these two points? [*Responses*: $\pi \subseteq X'X''$; $\underline{X'X'' \subseteq \pi}$; $X'X'' \cap \pi = \{X', X''\}$]
14. Is $N \in X'X''$? [*Responses*: *Yes*; No]
15. Therefore, would N be in π? [*Responses*: *Yes*, No]
16. But, in fact, *is* $N \in \pi$? [*Responses*: Yes; *No*]
17. Therefore, if $X \in s - \{N\}$, what can we say about the line NX? [*Responses*: NX is parallel to π; *NX intersects π in one and only one point*; NX may have two points in common with π for certain positions of X]
18. So, if $X \in s - \{N\}$, can we associate to this point *one and only one point* X' where $X' = NX \cap \pi$? [*Responses*: *Yes*; No]
19. Is St, therefore, fully determined by this rule? [*Responses*: *Yes*; No]

[*Audio*:] These last two arguments that you have gone through serve to bring out a very important point. Whenever we want to talk about a transformation from a set S to a set S', we must be sure that this transformation exists. That is, we must be sure that to each element in S we can actually make correspond *some* element in S', and we must be sure that each element in S corresponds to *no more* than one element in S'. In the geometric transformations we will be dealing with in our lessons, we will be able to check that our transformations exist and are well-defined by referring to various axioms and theorems of geometry that will guarantee their existence and well-defined character. In the case of stereographic projection, for example, we used properties about parallel planes, tangent planes, and incidence of lines and planes. Let's look briefly at a few more examples to reinforce these observations.

[Diagram above disappears, and diagram below appears now]

G: Gnomonic projection



[Audio:] Here the situation is rather like the one we were dealing with earlier in stereographic projection. As before, we have a sphere (lower case s) which is tangent to a plane π at the "south pole" (capital) S. O is the center of the sphere. Instead of projecting s onto π from N, however, we will project it from O. This transformation is called gnomonic projection, and we will denote it by capital G. To be more precise, if X is any point on s not on the "equator" e of s, then the point X' where the line OX intersects π we will call the gonomonic image of X under G, so that $G(X) = X'$.

[Questions (1)-(17) will appear in written form on the screen. Diagram on page 23 will remain through Question (17) below]

1. If E is on the "equator" e, what can we say about OE? [Responses: OE intersects π; OE is parallel to π; OE neither intersects nor is parallel to π]

2. If $OE \parallel \pi$, can OE intersect π? [*Responses*: Yes; *No*]
3. What is the "gnomonic image" of a point E on e? [*Responses*: E'; $G(E)$; *it doesn't exist*]
4. Now consider a point $X \in s - e$. Is $OX \parallel \pi$? [*Responses*: Yes; *No*]
5. Does OX intersect π in *at least one point*? [*Responses*: *Yes*; No; depends on the position of X]
6. If $X \in s - e$, can OX intersect π in two distinct points? [*Responses*: Yes; *No*; depends on the position of X]
7. If OX intersects π in two distinct points, what can we say about O? [*Responses*: $O = S$; $G(O) = O$; O is on s; *O is in π*]
8. But if a sphere is tangent to a plane, where does the center of the sphere lie? [*Responses*: On the plane; *Not on the plane*]
9. In short, if $X \in s - e$, in how many points does OX intersect π? [*Responses*: None; *exactly one*; at least one; two]
10. If $X \in s - e$, can we associate to this point *one and only one* point X' where $X' = OX \cap \pi$? [*Responses*: *Yes*; No]
11. Is G fully determined by this rule? [*Responses*: *Yes*; No]
12. Which geometric fact did we *not* use to verify that we actually have the existence of a well-defined transformation? [*Responses*: A line not parallel to a plane must intersect the plane; the center of a sphere tangent to a plane must lie in one of the two open half-spaces determined by the plane; *a sphere and a plane are tangent in at most one point*]

[Audio:] Here's a second situation, therefore, in which the existence and well-defined character of a transformation as defined by certain geometric instructions is guaranteed by axioms and theorems from geometry.

13. What is the domain of G? [Responses: s; e; $s - \{N\}$; $s - e$; $s - \{O\}$]

14. Are gnomonic images of points in $s - e$ points in π? [Responses: Yes; No]

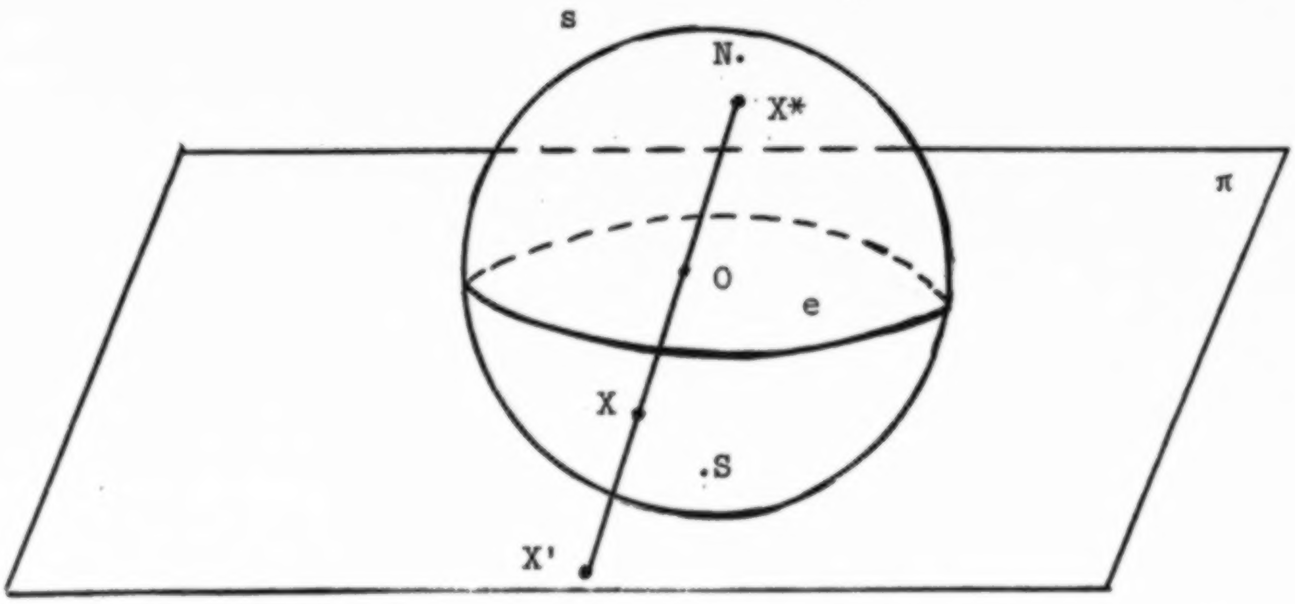
15. Is every point in π the gnomonic image of some point of $s - e$? [Responses: Yes; No]

16. Are you sure? [Responses: Yes, I could prove this as we did for St; No]

17. What is the range of G? [Responses: $s - e$; $\{S\}$; π; $\pi - \{S\}$]

[Diagram above disappears, and diagram below appears now]

[Audio:] Let's look at one further feature of this transformation for just a minute. Study the figure shown on the scope.



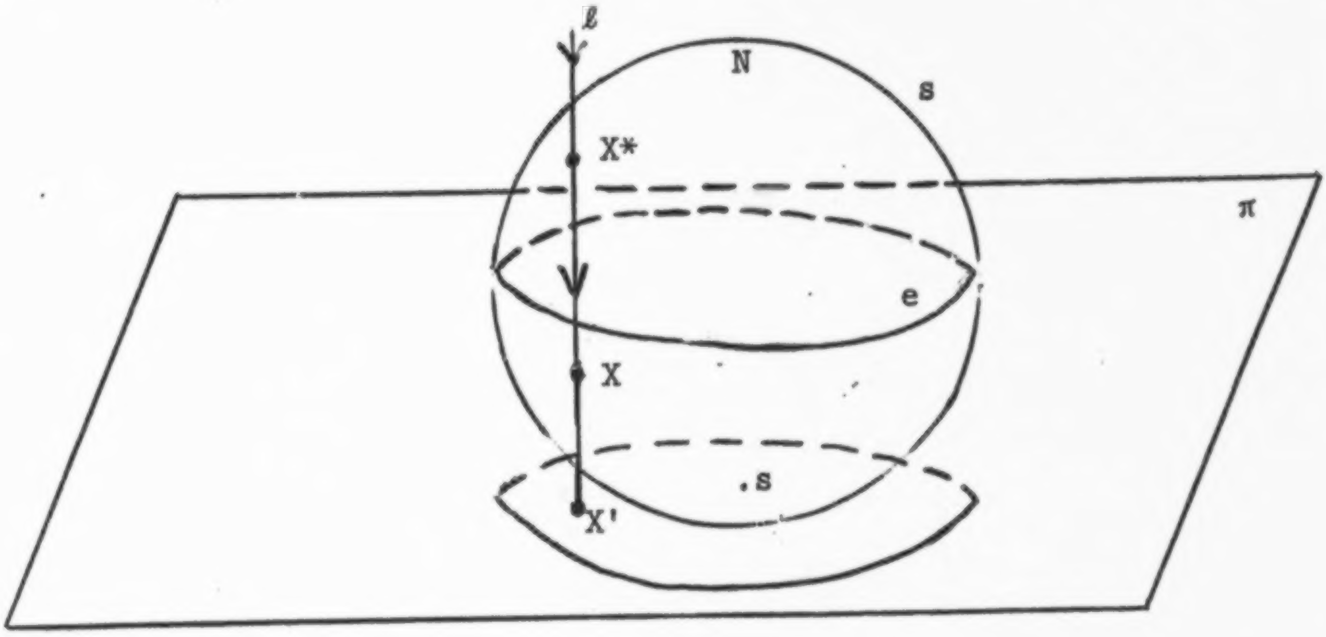
[Questions (1)-(12) will appear in written form on the screen. Diagram is to remain through Question 12 below]

1. From the diagram, what is $G(X)$? [Responses: X; X'; X^*]
2. Every line through the center of a sphere intersects the sphere in how many points? [Responses: One if the line is tangent to the sphere; always two]
3. In the diagram, what is $OX \cap s$? [Responses: $\{X\}$; $\{X^*\}$; $\{X, X^*\}$; $\{X'\}$]
4. What is $G(X^*)$? [Responses: X; X'; X^*]
5. Does $G(X) = G(X^*)$? [Responses: Yes; No]
6. For each $Z \in \pi$, how many points in s have Z as their gnomonic image? [Responses: exactly one; exactly two; depends on the position of Z]
7. Does stereographic projection St have this property? [Responses: Yes; No]
8. What is $G(N)$? [Responses: N; O; S]
9. Does $G(N) = G(O)$? [Responses: Yes; No; $G(O)$ is not defined]
10. Does $G(N) = G(S)$? [Responses: Yes; No]
11. What is $G(St(S))$? [Responses: N; O; S; $G(St(S))$ is undefined]
12. What is $St(G(N))$? [Responses: N; S; $St(G(N))$ is undefined]

[Diagram above disappears and diagram below appears now]

[Audio:] As a third example of a similar sort, suppose we have once again a sphere s tangent to a plane π at the "south pole" S. Let us project the sphere vertically downward to the plane π. This transformation is often called orthographic projection. We will denote it by O - one guess where we get O from. Again, to be more precise, if X is any point on s, then the point X' where the line ℓ through

O: *Orthographic projection*



X perpendicular to π intersects π we will call the *orthographic image* of X under O, so that O(X) = X'. Let's see you zip through these next questions now.

[Questions (1)-(23) will appear in written form on the screen. Diagram is to remain through Questions 23]

1. Why does the line ℓ through X perpendicular to π intersect π? [*Responses*: This is true of orthographic projection; *A line perpendicular to a plane always intersects that plane*; perpendicular distances are shortest distances]
2. Does ℓ intersect π in *at least one* point? [*Responses*: *Yes*; No]
3. Can ℓ intersect π in two distinct points? [*Responses*: Yes; *No*]
4. If ℓ were to intersect π in two distinct points, which of the following statements would *not* be true? [*Responses*: $X \in \pi$; $X^* \in \pi$; $\ell \subset \pi$; $\underline{\ell \perp \pi}$]
5. But is $\ell \perp \pi$? [*Responses*: *Yes*; No]

6. O.K., then, in how many points does ℓ intersect π? [*Responses*: None; *exactly one*; exactly two]

7. If $x \in s$, can we associate to this point *one and only one* point X', where $X' = \ell \cap \pi$? [*Responses*: *Yes*; No]

8. Is O fully determined by this rule? [*Responses*: *Yes*; No]

[*Audio*:] Orthographic projection gives us our third example of a situation in which the existence of a transformation is guaranteed by an appropriate definition. Here too we used certain elementary geometric facts to guarantee that a certain transformation exists and is well-defined.

9. What is the domain of O? [*Responses*: $\underline{s}$; $s - \{N\}$; $s - e$; $s - \{S\}$]

10. What is $O(X)$? [*Responses*: X; $\underline{X}'$; X^*]

11. What is $O(X^*)$? [*Responses*: X, $\underline{X}'$; X^*]

12. Does $O(X) = O(X^*)$? [*Responses*: *Yes*; No]

13. Is $X = X^*$? [*Responses*: Yes; No; *depends on the position of X*]

14. If $X = X^*$, what can you say about ℓ? [*Responses*: ℓ does not exist; ℓ is parallel to π; *ℓ is tangent to s*]

15. If ℓ is not tangent to s, in how many points does it intersect s? [*Responses*: 0; 1; *2*; 3]

16. Are orthographic images of points in s points in π? [*Responses*: *Yes*; No]

17. Is every point in π the orthographic image of some point of s? [*Responses*: Yes; *No*]

18. Suppose we can prove that O projects e onto a circle e'. If Z is a point in π outside this circle, how many points in s have Z as their orthographic image? [*Responses*: *0*; 1; 2]

19. If Z is a point in π on circle e', how many points in s have Z as their orthographic image? [*Responses*: 0; *1*; 2]

20. And if Z is a point in π inside e', how many points in s have Z as their orthographic image? [*Responses*: 0; 1; *2*]

21. What is the range of O? [*Responses*: π; e'; interior e'; exterior e'; *$e' \cup$ interior e'*]

22. What is O(G(N))? [*Responses*: N; O; *S*; O(G(N)) is not defined]

23. What is G(O(St(S)))? [*Responses*: N; O; *G(St(O(S)))*; G(O(St(S))) is not defined]

APPENDIX II

LESSON AS STORED IN COMPUTER

Text of Lesson I

Prepared by

Peter Belew

A transcription of Lesson I into the lesson language is given as an example and may be compared with the author's master text in Appendix I. There are some minor differences in wording and in the way questions have been presented.

The lesson language is described on page 58.

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<GEOMETRY LESSONS
<LESSON ONE:TOPIC 2A-- P.8.TOPIC 2B--P.15.TOPIC 3-P. 22
<LESSON TWO:TOPIC 0--P30.TOPIC 1--P37.TOPIC 2--P.51
<LESSON ONE:
(TOPIC   1)
(LOCSEC ( (1 7)(5 1)(6 1)(6 36) (6 46)(7 18)  )   )
(SECTION 1)
(DISPLAY 1)(PLAY 1 1 4)
(WAIT 10)
(DISPLAY 2)
(PLAY 2 1 5)
(CAPTION
/      A TRANSFORMATION FROM A SET S
INTO A SET S' IS A CORRESPONDENCE OF/)(CAPTION/
THE ELEMENTS OF ONE SET (S) WITH SOME OR/)(CAPTION/
ALL OF THE ELEMENTS OF A SECOND SET(S')/)(CAPTION/
IN SUCH A WAY THAT NO ELEMENT OF S IS/)(CAPTION/
MADE TO CORRESPOND TO MORE THAN ONE/)(CAPTION/
ELEMENT OF S'/)
(PLAY 3 1 8)(PLAY 4 1 4)(WAIT 10)(RESTORE)

(QUESTION 1    TRIALS 10)
(CAPTION/1.TWO ELEMENTS OF S ARE A AND D./)(CAPTION/
WHAT ARE THE REMAINING ELEMENTS OF S?
/)
(PLAY 5 1 4)
(ANSWERS SET)((B C E F ) RIGHT)(ELSE WRONG)(END)

(QUESTION 2)
(CAPTION /2.HOW MANY ELEMENTS ARE THERE
IN THE SET S'?/)(PLAY 6 1 3)(CAPTION/
(4,5,OR 6)/)
(ANSWERS)(4 WRONG (PLAY 7 1 4))
         (5 RIGHT)
(6 WRONG (PLAY 8 1 4))(END)

(QUESTION 3 I )
(CAPTION/3.I. THE ELEMENT A OF S CORRESPONDS WITH THE
ELEMENT H OF S'.  WHAT DOES THE ELEMENT B
OF S CORRESPOND WITH?  (G,H,I,J OR K)
/)(PLAY 9 1 5)
(ANSWERS)(G RIGHT)(H WRONG)(I WRONG)(J WRONG)(K WRONG)
(END)

(QUESTION 3 II)
(CAPTION /3.II.WHAT DOES THE ELEMENT C
OF S CORRESPOND WITH? ---F, I, OR K ?
/)(PLAY 10 1 2)
(ANSWERS)(F WRONG)(I WRONG)(K RIGHT)(END)
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(QUESTION 3 III )
(CAPTION/3.III.WHAT DOES THE ELEMENT D OF S
CORRESPOND WITH-----/)(PLAY 11 1 2)(CAPTION /
(1)A    (2)K
(3)THE SAME ELEMENT AS THAT TO WHICH B CORRESPONDS
(TYPE NUMBER)
/)
(ANSWERS)(1 WRONG)(2 WRONG)(3 RIGHT)(END)

(QUESTION 4 )
(CAPTION /4.ELEMENTS B AND D BOTH CORRESPOND TO
WHICH ELEMENT OF S'?  (G,H,I,J OR K)/)
(PLAY 12 1 3)
(ANSWERS)(G RIGHT)(H WRONG)(I WRONG)(J WRONG)(K WRONG)
(END)

(QUESTION 5 )
(CAPTION /5.IS THERE ANY ELEMENT IN S',OTHER THAN G,TO
WHICH TWO OR MORE ELEMENTS OF S CORRESPOND?(YES OR NO)
/) (PLAY 13 1 3)
(ANSWERS)(YES RIGHT)(NO WRONG)(END)

(QUESTION 6 )
(CAPTION/6.WHAT IS THIS ELEMENT,OTHER TNAN G, TO WHICH
TWO OR MORE ELEMENTS OF S CORRESPOND
--G,J,OR K?
/)(PLAY 14 1 2)
(ANSWERS)(G WRONG )(J WRONG)(K RIGHT)(END)

(QUESTION 7 )
(CAPTION/7.WHAT ARE THE ELEMENTS OF S CORRESPONDING
TO THIS ELEMENT K IN S'?/)(CAPTION/
(1)B AND D       (3)C AND F
(2)B AND F       (4)C ANC D
(TYPE NUMBER)        /)(PLAY 15 1 3)
(ANSWERS)(1 WRONG)(2 WRONG)(3 RIGHT)(4 WRONG)(END)

(QUESTION 8 )
(CAPTION /8.IS THERE ANY ELEMENT OF S
CORRESPONDING TO TWO ELEMENTS OF S'?
/)(PLAY 16 1 3)
(ANSWERS)(YES WRONG (PLAY 17 1 7))
 (NO RIGHT)(END)
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(QUESTION 9)
(CAPTION /9.SINCE NO ELEMENT OF S CORRESPONDS TO/)(CAPTION/
MORE THAN ONE ELEMENT IN S',WHAT/)(CAPTION/
DOES THE FIGURE ON THE SCOPE ILLUSTRATE?/)
(PLAY 18 1 4)    (CAPTION/
(1)A CORRESPONDENCE BETWEEN ALL THE
        ELEMENTS OF S AND ALL THE ELEMENTS OF S'/)(CAPTION/
(2)A TRANSFORMATION FROM THE SET S' INTO THE SET S/)(CAPTION/
(3)A TRANSFORMATION FROM THE SET S TO ALL THE ELEMENTS OF S'/)
(CAPTION/
(4)A TRANSFORMATION FROM S TO S'.       (TYPE NUMBER)
/)
(ANSWERS)(1 WRONG)(2 WRONG)(3 WRONG)(4 RIGHT)(END)

(QUESTION 10)
(CAPTION /10.IN THE TRANSFORMATION ILLUSTRATED ON THE
SCOPE, ARE THERE ANY ELEMENTS OF S' THAT ARE/)(CAPTION/
NOT CORRESPONDENTS OF ANY ELEMENTS
OF S? (YES OR NO)
/)(PLAY 19 1 5)
(ANSWERS)(YES RIGHT)(NO WRONG)(END)

(QUESTION 11)
(CAPTION/11.WHAT ELEMENTS OF S' ARE NOT CORRESPONDENTS
OF ANY ELEMENTS OF S?
(1)G       (3)J
(2)I       (4)K
     (5)G AND K      (TYPE NUMBER)
/)(PLAY 20 1 2)
(ANSWERS)(J RIGHT)(1 WRONG)(2 WRONG)(3 RIGHT)(4 WRONG)
         (5 WRONG)(END)

(QUESTION 12)
(CAPTION/12.FROM THE EXAMPLE SHOWN ON THE SCOPE, WOULD
YOU SAY THAT IF WE HAVE A TRANSFORMATION FROM A SET S/)
(CAPTION/
TO A SET S',THEN EVERY ELEMENT OF S' MUST BE THE
CORRESPONDENT OF SOME ELEMENT OF S?     (YES OR NO)
/)  (PLAY 21 1 6)
(ANSWERS)(YES WRONG)(NO RIGHT)(END)

(QUESTION 13)
(CAPTION/13.AND IF WE HAVE A TRANSFORMATION FROM S TO
S', MUST ELEMENTS OF S' BE CORRESPONDENTS OF
EXACTLY ONE ELEMENT OF S?
/)(CAPTION/TYPE YES OR NO
/)(PLAY 22 1 5)
(ANSWERS)(YES WRONG)(NO RIGHT)(END)
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(QUESTION  14)
(CAPTION/14.TO BE A TRANSFORMATION, HOWEVER,
EACH ELEMENT OF S MUST CORRESPOND TO
HOW MANY  ELEMENTS OF S'/)(PLAY 23 1 4)(CAPTION/
(1) AT LEAST ONE       (3) SOMETIMES TWO
(2) SOMETIMES NONE      (4) EXACTLY ONE    (TYPE NUMBER)
/)
(ANSWERS)(1 WRONG)(2 WRONG)(3 WRONG)(4 RIGHT)(END)

(ACTION (1)(8 16) 1)

(SECTION 2)

(DISPLAY 3) (WAIT 5)
(CAPTION/LET'S TRY SOME PROBLEMS TO SEE IF THE
IDEA OF A TRANSFORMATION IS CLEAR TO YOU
/)(PLAY 24 1 5)(WAIT 5)(RESTORE)

(QUESTION 1)
(CAPTION/1.   WHICH OF THESE DIAGRAMS ILLUSTRATE A
TRANSFORMATION FROM S TO S'?/)(PLAY 25 1 3)
(CAPTION/ANSWERS:
(A)1,2 AND 3          (D)1 AND 3
(B)1 AND 2            (E)1
(C)2 AND 3       (TYPE LETTER)
/)
(ANSWERS)(A WRONG)(B WRONG)(C WRONG)(D RIGHT)
         (E WRONG)(END)

(QUESTION 2 )
(CAPTION/2. DIAGRAM 2 DOES NOT ILLUSTRATE A TRANSFORMATION
SINCE THERE IS AN ELEMENT IN S WHICH CORRESPONDS TO/)(CAPTION/
MORE THAN ONE ELEMENT IN S'.WHICH ELEMENT IS THIS?
..A,B,C OR F ?/)(PLAY 26 1 6)
(ANSWERS)(A WRONG)(B RIGHT)(C WRONG)(F WRONG)(END)

(QUESTION 3 )
(CAPTION /3. IN DIAGRAM 2,WHAT ARE THE TWO ELEMENTS IN S'
TO WHICH THE ELEMENT B CORRESPONDS?     (TYPE NUMBER)
(1)D AND E
(2)E AND F
(3)D AND F          /)
(PLAY 27 1 4)
(ANSWERS)(1 WRONG)(2 WRONG)(3 RIGHT)(END)
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(QUESTION 4 )
(CAPTION/4.FOR A CORRESPONDENCE FROM A SET S TO A SET
S' TO BE A TRANSFORMATION, HOWEVER, EACH ELEMENT OF/)
(CAPTION/
S MUST CORRESPOND TO EXACTLY HOW MANY ELEMENTS IN S'?
(1)ONE
(2)TWO
(3)THE ELEMENT E OF S'  (TYPE 1,2 OR 3)
/)(PLAY 28 1 6)
(ANSWERS)(1 RIGHT)(2 WRONG)(3 WRONG)(END)

(SECTION  3)
(CAPTION/ANSWER THE FOLLOWING QUESTIONS./)
(PLAY 29 1 2)(WAIT 5)(RESTORE)

(QUESTION 5 I  TIME 30)
(CAPTION/5.I.        IN DIAGRAM 1, ARE THERE ANY ELEMENTS OF
S' THAT ARE NOT CORRESPONDENTS OF ELEMENTS OF S?
(ANSWER YES OR NO)  /)
(ANSWERS)(YES WRONG)(NO RIGHT)(END)

(QUESTION 5 II  TIME 30 )
(CAPTION /5.II.      IN WHICH OF THE FOLLOWING WAYS COULD
DIAGRAM 2 BE ALTERED TO ILLUSTRATE A TRANSFORMATION?/)
(CAPTION/
(1)CONNECT A TO D BY AN ARROW
(2)REMOVE ONE OF THE ARROWS POINTING TO E
(3)REMOVE THE ARROW POINTING TO D.      (TYPE NUMBER)
/)       (ANSWERS)(1 WRONG)(2 WRONG)(3 RIGHT)(END)

(QUESTION 5  III TIME 30)
(CAPTION /5.III. IN DIAGRAM 3, HOW MANY ELEMENTS
OF S CORRESPOND TO E? (0,1,2 OR 3)/)
(ANSWERS)(0 WRONG)(1 WRONG)(2 WRONG)(3 RIGHT)(END)

(QUESTION 5 IV TIME 30)
(CAPTION /5.IV. IN DIAGRAM 3, HOW MANY ELEMENTS OF S
CORRESPOND TO F? (0,1,2 OR 3)/)
(ANSWERS)(0 RIGHT)(1 WRONG)(2 WRONG)(3 WRONG)(END)

(QUESTION 5 V TIME 30)
(CAPTION/5.V. HOW MANY ELEMENTS OF THE SET
S' IN DIAGRAM 3 ARE CORRESPONDENTS OF
ELEMENTS OF S? (0,1,2 OR 3)/)
(ANSWERS)(0 WRONG)(1 RIGHT)(2 WRONG)(3 WRONG)(END)
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(SECTION 4)

(QUESTION 5 VI TIME 30)
(CAPTION /5.VI. DOES DIAGRAM 3 ILLUSTRATE A TRANSFORMATION?
        (YES OR NO)    /)
(ANSWERS)(YES RIGHT)(NO WRONG)(END)

(ACTION (4)(2 10) 5)
(GOTO 6)

(SECTION 5)
(PLAY 30 1 8)       <THINK ABOUT THIS

(QUESTION 5 VI1 TIME 15)
(CAPTION/5.VI.1. WHAT IS THE ONE ELEMENT OF S' TO WHICH
A CORRESPONDS (TYPE D, E OR F)  /)
(ANSWERS)(D WRONG )(E RIGHT)(F WRONG)(END)

(QUESTION 5 VI2 TIME 15)
(CAPTION/5.VI.2 WHAT IS THE  ONE ELEMENT OF S' TO WHICH
B CORRESPONDS?       /)
(ANSWERS)(D WRONG)(E RIGHT)(F WRONG)(END)

(QUESTION 5 VI3 TIME 15)
(CAPTION /5.VI.3. WHAT IS THE ONE ELEMENT OF S' TO WHICH
C CORRESPONDS?       /)
(ANSWERS)(D WRONG)(E RIGHT)(F WRONG)(END)

(PLAY 31 1 8)

(SECTION 6)
(ACTION (3 4)(6 8)3)
(ACTION (1 2 3 4)(14 50) 2)
(FINIS)

(TOPIC 2 A)
(LOCSEC((8 4)(11 3)(12 1)(13 1)(13 9)(13 20) ) )

(SECTION 1)
(DISPLAY 5)(PLAY 32 1 3)    <TOPIC 2 HEADING
(WAIT 2)(PLAY 33 1 7)(WAIT 2)
(DISPLAY 6)
(CAPTION/WE SAY BY DEFINITION THAT THE DOMAIN OF
A TRANSFORMATION IS THE SET OF ALL/)(CAPTION/
ELEMENTS OF THE FIRST SET THAT COR-
RESPOND TO SOME ELEMENT IN THE SEC-/)(CAPTION/
OND SET./)(PLAY 34 1 5)
(CAPTION/IN TERMS OF THE DIAGRAMS
WE HAVE BEEN USING,AN OBJECT WOULD/)(CAPTION/
BE AN ELEMENT OF THE DOMAIN OF A
TRANSFORMATION WHEN THERE IS AN/)(CAPTION/
ARROW LEADING FROM THIS ELEMENT./)
(PLAY 35 1 5)
(WAIT 15)(RESTORE)
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(QUESTION 1)<WHAT IS THE DOMAIN?
(PLAY 36 1 3)
(CAPTION /1.
(1)S
(2)S'
(3)(E,F,G,I)  (TYPE NUMBER)      /)
(ANSWERS)(1 RIGHT)(2 WRONG)(3 WRONG)(END)

(QUESTION 2)<IS THERE AN ARROW LEADING FROM C?
(PLAY 37 1 3)
(CAPTION/2.
TYPE YES OR NO      /)
(ANSWERS)(YES RIGHT)(NO WRONG)(END)

(QUESTION 3)<IS C AN ELEMENT OF THE DOMAIN?
(PLAY 38 1 3)
(CAPTION /3.       /)
(ANSWERS)(YES RIGHT)(NO WRONG)(END)

(QUESTION 4)<IS THERE AN ARROW LEADING FROM E?
(PLAY 39 1 2)
(CAPTION /4.     /)
(ANSWERS)(YES WRONG)(NO RIGHT)

(QUESTION 5)<IS E AN ELEMENT OF THE DOMAIN?
(PLAY 40 1 3)
(CAPTION /5.      /)
(ANSWERS)(YES WRONG)(NO RIGHT)(END)

(QUESTION 6)
(CAPTION /6.      /)  (PLAY 41 1 4)
<WHICH OF THE FOLLOWING SETS
<CONTAINS NO ELEMENT OF THE DOMAIN
<THIS TRANSFORMATION?
(CAPTION /(TYPE NUMBER)
(1)(A,D,F,G)     (3)(B,C,D,H)
(2)(H,I,A,B)     (4)(F,G,H,I)
/)
(ANSWERS)(1 WRONG)(2 WRONG)(3 WRONG)(4 RIGHT)(END)

(QUESTION 7)
(CAPTION /7.    /)(PLAY 42 1 4)
<WHAT IS THE SET OF ELEMENTS THAT COMPRISES THE DOMAIN
<OF THIS TRANSFORMATION?
(CAPTION /
(1)(A,B,C,D)
(2)(E,F,G,I)
(3)(E,F,G,H,I)       /)
(ANSWERS)(1 RIGHT)(2 WRONG)(3 RIGHT)(END)
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(CAPTION /    THE RANGE OF A TRANSFORMATION, ON THE
OTHER HAND, IS THE SET OF ALL
ELEMENTS OF THE SECOND SET THAT/)(CAPTION/
CORRESPOND TO ONE OR MORE ELE-
MENTS OF THE FIRST SET./)(PLAY 43 1 5)(CAPTION/
   IN TERMS OF OUR SCHEMATIC
DIAGRAM,AN OBJECT WOULD BE AN/)(CAPTION/
ELEMENT OF THE RANGE OF A TRANS-
FORMATION WHEN THERE IS AN ARROW
LEADING TO THIS ELEMENT./)
(PLAY 44 1 5)
(WAIT 20)(RESTORE)

(QUESTION  8)
(CAPTION /8.     /)
<ALL ELEMENTS OF THE RANGE OF THE
<TRANSFORMATION SHOWN BELONG TO WHICH SET---
(PLAY 45 1 4)(CAPTION/
(1)S, OR
(2)S'    (TYPE NUMBER)  /)
(ANSWERS)(1 WRONG)(2 RIGHT)(END)

(QUESTION 9)
(CAPTION /9. /)
<IS THERE AN ARROW LEADING TO THE ELEMENT
<E IN S'?
(PLAY 46 1 3)
(ANSWERS)(YES RIGHT)(NO WRONG)(END)

(QUESTION 10)
(CAPTION /10./)
<IS E IN THE RANGE OF THIS TRANSFORMATION?
(PLAY 47 1 3)
(ANSWERS)(YES RIGHT)(NO WRONG)(END)

(QUESTION 11)
(CAPTION /11.  /)
<(COMMENT IS D IN THE RANGE OF THIS TRANSFORMATION?)
(PLAY 48 1 3)
(ANSWERS)(YES WRONG)(NO RIGHT)(END)

(QUESTION 12)
(CAPTION /12.  /)
< HOW MANY ARROWS LEAD TO H?
(PLAY 49 1 2)(CAPTION /
(TYPE 0,1,2 OR 3)   /)
(ANSWERS)(0 RIGHT)(1 WRONG)(2 WRONG)(3 WRONG)(END)

(QUESTION 13)
(CAPTION /13.  /)
(COMMENT  IS H IN THE RANGE OF THIS TRANSFORMATION?)
(PLAY 50 1 3)
(ANSWERS)(YES WRONG)(NO RIGHT)(END)
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(QUESTION 14)
(CAPTION /14./)
< WHAT IS THE RANGE OF THIS TRANSFORMATION?--
(PLAY 51 1 2)(CAPTION/
(A)S          (C)(H)
(B)S'         (D)NONE OF THESE    (TYPE LETTER)
/)
(ANSWERS)(A WRONG)(B WRONG)(C WRONG)(D RIGHT)(END)

(QUESTION 15)
(CAPTION /15. /)
< WHAT IS THE SET OF ELEMENTS THAT COMPRISES
<THE RANGE OF THIS TRANSFORMATION?--
(PLAY 52 1 3)
(CAPTION /(TYPE LETTER)
(A)S                (C)(E,F,G,I)
(B)(A,B,C)          (D)(E,F,G,H)
/)
(ANSWERS)(A WRONG)(B WRONG)(C RIGHT)(D WRONG)(END)

(ACTION (1) (10 16) 1)

(PLAY 53 1 6)

(SECTION 2)
(DISPLAY 7)
(QUESTION 1     TRIALS 10)
(CAPTION /1. WHAT IS THE DOMAIN OF THIS TRANSFORMATION?
 (TYPE ELEMENTS)/)
(ANSWERS SETS)((A,B,C,D)RIGHT )(ELSE WRONG)(END)

(QUESTION 2)
(CAPTION/2. WHAT IS THE RANGE?
(1)S          (3)(E)
(2)S'         (4)(F)  (TYPE NUMBER)
/)
(ANSWERS)(1 WRONG)(2 WRONG)(3 WRONG)(4 RIGHT)(END)

(QUESTION 3)
(CAPTION/3. IS THERE AN ARROW LEADING TO G?
/)
(ANSWERS)(YES WRONG)(NO RIGHT)(END)

(QUESTION 4)
(CAPTION /4. IS G IN THE RANGE?
/)
(ANSWERS)(YES WRONG)(NO RIGHT)(END)
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(QUESTION 5)
(CAPTION /5. WHICH OF THE FOLLOWING SETS CONSISTS OF JUST
THOSE ELEMENTS OF S' WHICH ARE NOT IN THE RANGE
OF THIS TRANSFORMATION?
(1)(F)          (3)(E,G,H)
(2)(G,H)        (4)(A,B,C,D)
/)
(ANSWERS)(1 WRONG)(2 WRONG)(3 RIGHT)(4 WRONG)(END)

(QUESTION 6)
(CAPTION /6. IF WE HAVE A TRANSFORMATION FROM A SET S TO A
SET S', DOES THE RANGE ALWAYS HAVE TO BE THE SET S'?
/)
(ANSWERS)(YES WRONG)(NO RIGHT)(END)

(ACTION (2)(6 8) 2)

(SECTION 3)
(DISPLAY 8)

(QUESTION 1)
(CAPTION /1. WHAT ARE THE ELEMENTS OF THE SET S?
/)
(ANSWERS SETS)((A B C D) RIGHT)(ELSE WRONG)(END)

(QUESTION 2)
(CAPTION /2. HOW MANY ARROWS LEAD FROM A?(TYPE 0,1 OR 2)
/)
(ANSWERS)(/1/RIGHT)(/2/WRONG)(/0/WRONG)(END)

(QUESTION 3)
(CAPTION /3. HOW MANY ARROWS LEAD FROM B?(TYPE 0,1,2 OR 3)
/)
(ANSWERS)(/0/WRONG)(/1/RIGHT)(/2/WRONG)(/3/WRONG)(END)

(QUESTION 4)
(CAPTION /4. HOW MANY ARROWS LEAD FROM C?(0,1,2 OR 3)
/)
(ANSWERS)(0 RIGHT)(1 WRONG)(2 WRONG)(3 WRONG)(END)

(QUESTION 5)
(CAPTION /5. HOW MANY LETTERS LEAD FROM D?(0,1,2 OR 3)
/)
(ANSWERS)(0 WRONG)(1 RIGHT)(2 WRONG)(3 WRONG)(END)

(QUESTION 6)
(CAPTION /6. WHICH OF THE FOLLOWING IS THE SET OF ALL ELEMENTS
OF S THAT HAVE EXACTLY ONE ARROW LEADING FROM THE ELEMENT?
(1) (A,B,C,D)
(2) (A,B,D)
(3) (A,C,D)      /)
(ANSWERS)(1 WRONG)(2 RIGHT)(3 WRONG)(END)
```

```
(SECTION 4)
(SET 1)
(QUESTION 7)
(CAPTION /10. CONSIDER THE SET (A,B,D). DO WE HAVE A
TRANSFORMATION FROM THIS SET TO S'?
/)
(ANSWERS)(YES RIGHT)(NO WRONG (UNSET 1))(END)
(IF 1 THEN GOTO 6)
(SECTION 5 )
         (QUESTION 7 A)(PLAY 54 1 4)
<LOOK AGAIN. HOW MANY ARROWS LEAD OUT OF EACH
<OF THE ELEMENTS A,B AND D?
(CAPTION/
(1)AT LEAST ONE
(2)EXACTLY ONE
(3)SOMETIMES ONE
/)
(ANSWERS)(1 WRONG)(2 RIGHT)(3 WRONG)(END)
(PLAY 55 1 2)(GOTO 10)
 (SECTION  6)
(QUESTION 8)
(CAPTION /8. WHAT IS THE DOMAIN OF THIS TRANSFORMATION?
(1)S                (3)(A,B,D)
(2)(A,B,C)          (4)(C)
         (5)S'
/)
(ANSWERS)(1 WRONG)(2 WRONG)(3 RIGHT)(4 WRONG)
         (5 WRONG)(END)

(QUESTION 9)
(CAPTION /9. IS THE DOMAIN OF THIS TRANSFORMATION THE
SAME AS THE SET S?        /)
(ANSWERS)(YES WRONG)(NO RIGHT)(END)

(QUESTION 10)
(CAPTION /10. DO WE HAVE A TRANSFORMATION FROM THE SET
S TO THE SET S'?       /)
(ANSWERS)(YES WRONG)(NO RIGHT)(END)

(QUESTION 11)
(CAPTION /11. BUT DO WE HAVE A TRANSFORMATION FROM A
SUBSET OF S TO S'?        /)
(ANSWERS)(YES RIGHT)(NO WRONG)(END)

(QUESTION 12)
(CAPTION /12. WHICH OF THE FOLLOWING IS THIS SUBSET OF S?
(1)S                (3)(C)
(2)(B,C,D)          (4)(A,B,D)
        (5)THE EMPTY SET
/)
(ANSWERS)(1 WRONG)(2 WRONG)(3 WRONG)(4 RIGHT)
         (5 WRONG)(END)

(PLAY 57 1 7)(PLAY 57 1 8)
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(QUESTION 13)
(CAPTION /13. TYPE ALL THE ELEMENTS OF THE RANGE OF THE
TRANSFORMATION WITH DOMAIN (A,B,D)!
/)
(ANSWERS SET)((E F G)RIGHT)(ELSE WRONG)(END)

(QUESTION 14)
(CAPTION /14. IS THE RANGE ALL OF S'?
/)
(ANSWERS)(YES WRONG)(NO RIGHT)(END)

(QUESTION 15)
(CAPTION /15. WHICH ELEMENT OF S' IS NOT IN THE RANGE
OF THIS TRANSFORMATION?
/)
(ANSWERS)(E WRONG)(F WRONG)(G WRONG)(H  RIGHT)(END)

(ACTION(3 4 5 6)(10 16)3)
(FINIS)

(TOPIC 2 B)
(LOCSEC((15 3)(16 1)(17 1)(18 1)(20 1)(21 38)))
(SECTION 1)(DISPLAY 9)

(QUESTION 1   TRIALS 10)
(CAPTION /1.  TYPE THE ELEMENTS OF THE SET S!
/)
(ANSWERS SETS)((A B C D) RIGHT)(ELSE WRONG)(END)

(QUESTION 2)
(CAPTION /2. IS G IN S?
/)
(ANSWERS)(YES WRONG)(NO RIGHT)(END)

(QUESTION 3)
(CAPTION /3. IS THERE EXACTLY ONE ARROW LEADING FROM B?
/)
(ANSWERS)(YES RIGHT)(NO WRONG)(END)

(QUESTION 4)
(CAPTION /4. IS THERE EXACTLY ON  ARROW LEADING TO H?
/)
(ANSWERS)(YES WRONG)(NO RIGHT)(END)

(QUESTION 5)
(CAPTION /5. DOES THE DIAGRAM ILLUSTRATE A TRANSFORMATION?
/)
(ANSWERS)(YES RIGHT)(NO WRONG)(END)
```

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(QUESTION 6)
(CAPTION /6. WHICH OF THE FOLLOWING IS THE DOMAIN OF THE
TRANSFORMATION?
(1)S
(2)S'
(3)(E,G,H)        /)
(ANSWERS)(1 RIGHT)(2 WRONG)(3 WRONG)(END)

(QUESTION 7)
(CAPTION /7. WHICH OF THE FOLLOWING IS THE RANGE OF THIS
TRANSFORMATION?
(1)S            (3)(E,G,H)
(2)S'           (4)(F)
/)
(ANSWERS)(1 WRONG)(2 WRONG)(3 RIGHT)(4 WRONG)(END)

(QUESTION 8)
(CAPTION /8. IS THIS TRANSFORMATION
(1)FROM S TO S',
(2)FROM S' TO S, OR
(3)FROM (E,G,H) TO S? (TYPE NUMBER)
/)
(ANSWERS)(1 RIGHT)(2 WRONG)(3 WRONG)(END)

(ACTION (1)(6 10) 1)

(SECTION 2)(DISPLAY 10)

(QUESTION 1)
(CAPTION /1. WHICH OF THE FOLLOWING IS THE SET S'?
(1)(A,B,C,D)       (3)THE RANGE OF THE TRANSFORMATION
(2)(E,G,H)         (4)(E,F,G,H)
/)
(ANSWERS)(1 WRONG)(2 WRONG)(3 WRONG)(4 RIGHT)(END)

(QUESTION 2)
(CAPTION /2. IS F IN (1) S OR (2) S'? (TYPE NUMBER)?
/)
(ANSWERS)(1 WRONG)(2 RIGHT)(END)

(QUESTION 3)
(CAPTION /3. IS THERE EXACTLY ONE ARROW LEADING FROM D?
/)
(ANSWERS)(YES RIGHT)(NO WRONG)(END)

(QUESTION 4)
(CAPTION /4. IS THERE EXACTLY ONE ARROW LEADING FROM
EACH OF A, B, AND C?
/)
(ANSWERS)(YES RIGHT)(NO WRONG)(END)
```

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(QUESTION 5)
(CAPTION /5. DO WE HAVE A TRANSFORMATION ILLUSTRATED BY THE
DIAGRAM?              /)
(ANSWERS)(YES RIGHT)(NO WRONG)(END)

(QUESTION 6)
(CAPTION /6. WHAT IS THE DOMAIN OF THIS TRANSFORMATION?
(1)S            (3)(B,D)
(2)S'           (4)(E,G,H)
/)
(ANSWERS)(1 RIGHT)(2 WRONG)(3 WRONG)(4 WRONG)(END)

(QUESTION 7)
(CAPTION /7. WHAT ELEMENT IS IN NEITHER THE DOMAIN NOR
THE RANGE OF THIS TRANSFORMATION?(TYPE B,D,F OR H)
/)
(ANSWERS)(B WRONG)(D WRONG)(F RIGHT)(H WRONG)(END)

(QUESTION 8)
(CAPTION /8. IS THIS A TRANSFORMATION
(1)FROM S TO S',
(2)FROM S' TO S,OR
(3)FROM (E,G,H) TO S?   (TYPE NUMBER)
/)
(ANSWERS)(1 RIGHT)(2 WRONG)(3 WRONG)(END)

(ACTION (2)(6 10) 2)

(SECTION 3)(DISPLAY 11)

(PLAY 58 1 4)

(QUESTION 1 TIME 30)(PLAY 59 1 3)
(CAPTION /1. THE DOMAIN IS:
(1)S
(2)(E,G,H)
(3)S'         /)
(ANSWERS)(1 RIGHT)(2 WRONG)(3 WRONG)(END)

(QUESTION 2 TIME 30)(PLAY 60 1 3)
(CAPTION /2. THE RANGE IS:
(1)S
(2)(E,G,H)
(3)S'          /)
(ANSWERS)(1 WRONG)(2 RIGHT)(3 WRONG)(END)

(QUESTION 3 TIME 30) (PLAY 61 1 4)
(CAPTION /3. WITH WHAT ELEMENT OF S' DOES A CORRESPOND?
(TYPE E,F,G OR H)       /)
(ANSWERS)(E RIGHT)(F WRONG)(G WRONG)(H WRONG)(END)
```

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(QUESTION 4 TIME 30)(PLAY 62 1 3)
(CAPTION /4. IN BOTH TRANSFORMATIONS, WITH WHAT ELEMENT
DOES B CORRESPOND? (TYPE E,F,G,OR H)
/)
(ANSWERS)(E WRONG)(F WRONG)(G WRONG)(H RIGHT)(END)

(QUESTION 5 TIME 30)(PLAY 63 1 2)
(CAPTION /5. WITH WHAT ELEMENT DOES C CORRESPOND?
(E,F,G OR H)          /)
(ANSWERS)(E WRONG)(F WRONG)(G RIGHT)(H WRONG)(END)

(QUESTION 6 TIME 30)(PLAY 64 1 1)
(CAPTION /6.AND D?        /)
(ANSWERS)(E WRONG)(F WRONG)(G WRONG)(H RIGHT)(END)

(QUESTION 7 TIME 30)(PLAY 65 1 1)
(CAPTION /7. HOW ABOUT E?
(1)A
(2)CANNOT BE DETERMINED
(3)E IS NOT IN THE DOMAIN OF EITHER TRANSFORMATION
(TYPE NUMBER)/)
(ANSWERS)(1 RIGHT)(2 WRONG)(3 RIGHT)(END)

(PLAY 66 1 7)
(QUESTION 8 TIME 30)(PLAY 67 1 3)
(CAPTION /8. WOULD IT MAKE ANY DIFFERENCE WHERE
THE SETS S AND S' ARE REPRESENTED?
/)
(ANSWERS)(YES WRONG)(NO RIGHT)(END)

(ACTION (3) (6 10) 3)

(SECTION 4)(DISPLAY 12)(PLAY 68 1 7)(PLAY 69 1 5)(PLAY 70 1 8)
<AUDIO IS RECORDED FOR QUES. 1-10, BUT DON'T USE!
(QUESTION 1)
(CAPTION /1. WHICH OF THE FOLLOWING IS AN ELEMENT OF BOTH
S AND S'?   (A,B,F,D)
/)
(ANSWERS)(A WRONG)(B RIGHT)(F WRONG)(D WRONG)(END)

(QUESTION 2)
(CAPTION /2. WHAT IS THE DOMAIN OF THE TRANSFORMATION
INDICATED IN THE FIGURE?
(1)S          (3)(A,B,E)
(2)(G,I)      (4)(C)
     (5)(C,F,G,I)
/)
(ANSWERS)(1 RIGHT)(2 WRONG)(3 WRONG)(4 WRONG)
          (5 WRONG)(END)
```

```
(SECTION 5)(DISPLAY 13)
(PLAY 81 1 7)
(QUESTION 1 TRIALS 10)
(CAPTION /1. THE DIAGRAM ILLUSTRATES A TRANSFORMATION FROM
A SET S TO A SET S'.TYPE THE ELEMENTS OF THE SET S!
/)
(ANSWERS SETS)((A B C D E) RIGHT)(ELSE WRONG)(END)

(QUESTION 2)
(CAPTION /2. WHICH OF THE ELEMENTS OF S DOES NOT HAVE AN ARROW
LEADING FROM IT?
(1)D
(2)C AND D
(3)THEY ALL DO.      /)
(ANSWERS)(1 WRONG)(2 WRONG)(3 RIGHT)(END)

(QUESTION 3)
(CAPTION /3. WHAT IS THE DOMAIN OF THIS TRANSFORMATION?
(1)THE EMPTY SET
(2)(A,B,C,E)
(3)(D)
(4)(A,B,C,D,E)   (TYPE NUMBER)
    /)
(ANSWERS)(1 WRONG)(2 WRONG)(3 WRONG)(4 RIGHT)(END)

(QUESTION 4)
(CAPTION /4. HOW MANY ELEMENTS ARE THERE IN S'?
(1,2,3,4 OR 5)        /)
(ANSWERS)(1 WRONG)(2 WRONG)(3 WRONG)(4 WRONG)
         (5 RIGHT)(END)

(QUESTION 5)
(CAPTION /5. WHAT IS THE RANGE OF THIS TRANSFORMATION?
(1)(A,B,C,D)       (3)(D)
(2)S               (4)NONE OF THESE.
/)
(ANSWERS)(1 WRONG)(2 RIGHT)(3 WRONG)(4 WRONG)(END)

(QUESTION 6)
(CAPTION /6. ARE S AND S' THE SAME SET HERE?
/ )
(ANSWERS)(YES RIGHT)(NO WRONG)(END)

(QUESTION 7)
(CAPTION /7.ARE THE DOMAIN AND RANGE OF THIS TRANSFORMATION
THE SAME SET?       /)
(ANSWERS)(YES RIGHT)(NO WRONG)(END)
```

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(QUESTION 8)
(CAPTION /8. IS THERE ANY ELEMENT OF S THAT CORRESPONDS
TO ITSELF?            /)
(ANSWERS)(YES RIGHT)(NO WRONG)(END)

(QUESTION 9)
(CAPTION /9. WHICH OF THE FOLLOWING IS IT?
(A,B,C,D,E)       /)
(ANSWERS)(A WRONG)(B WRONG)(C WRONG)(D RIGHT)(E WRONG)(END)

(QUESTION 10)
(CAPTION /10. TO WHAT ELEMENT DOES THE ELEMENT,TO WHICH C
CORRESPONDS, CORRESPOND? (TYPE A,B,C,D,OR E)
/)
(ANSWERS)(A WRONG)(B WRONG)(C WRONG)
         (D WRONG)(E RIGHT)(END)
(IF RIGHT THEN GOTO 6)
(COMMENT AUDIO MESSAGE)

(QUESTION 10 I)(PLAY 82 1 4)
(CAPTION /10.I. WHAT IS THE ELEMENT TO WHICH C CORRESPONDS?
(A,B,C OR E)       /)
(ANSWERS)(A WRONG )(B RIGHT)(C WRONG)(E WRONG)(END)

(QUESTION 10 II)
(CAPTION /10.II. SO NOW, TO WHAT ELEMENT DOES B CORRESPOND?
(A,B,C,D OR E)     /)(PLAY 83 1 3)
(ANSWERS)(A WRONG)(B WRONG)(C WRONG)(D WRONG)(E RIGHT)(END)

(QUESTION 10 III)
(CAPTION /10.III. THEREFORE, TO WHAT ELEMENT DOES THE ELEMENT,
TO WHICH C CORRESPONDS, CORRESPOND?(A,B,C,D, OR E?)
/)(PLAY 84 1 4)
(ANSWERS)(A WRONG)(B WRONG)(C WRONG)(D WRONG)(E RIGHT)(END)

(SECTION 6)
(ACTION (5)(8 12)5)
(ACTION (1 2 3 4 5)(21 30) 1)
(FINIS)

(TOPIC 3)(LOCSEC((22 2)(24 1)(26 1)(27 1)))
(SECTION 1)
(DISPLAY 15)
(PLAY 85 1 3)(WAIT 3)
(DISPLAY 16)<FIRST PICTURE
(PLAY 86 1 3)
(PLAY 87 1 5)
(DISPLAY 17)  <FIRST TRANSFORMATION
(PLAY 88 1 3)
(DISPLAY 18)  <SECOND TRANSFORMATION
(WAIT 3)
(PLAY 89 1 6)
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(DISPLAY 17)(WAIT 2)(DISPLAY 18)
(PLAY 90 1 4)
(DISPLAY 19)  <SIDE BY SIDE
(PLAY 91 1 6)
(DISPLAY 21)  <WITH DIMMED ARROWS
(PLAY 92 1 8)
(PLAY 93 1 8)
(PLAY 94 1 4)(PLAY 95 1 6)
(DISPLAY 22)
(PLAY 96 1 8)(WAIT 7)
(DISPLAY 23)
(PLAY 97 1 7)(WAIT 10)
(DISPLAY 24)
(PLAY 98 1 3)        <ADDITIONS UNNECESSARY!

<QUESTIONS FOR SEC. 1

(QUESTION 1)
(CAPTION /1. SUPPOSE WE DENOTE THIS TRANSFORMATION BY T.
WHAT IS T(A)?    (G,H,I,J OR K?)
/)
(ANSWERS)(G RIGHT)(H WRONG)(I WRONG)(J WRONG)(K WRONG)(END)

(QUESTION 2)
(CAPTION /2. HOW WOULD YOU WRITE THE ELEMENT TO
WHICH B CORRESPONDS?
(1)B
(2)H
(3)T(B)
(4)T(I)             /)
(ANSWERS)(1 WRONG)(2 WRONG)(3 RIGHT)(4 WRONG)(END)

(QUESTION 3)
(CAPTION /3. DOES T(A)=T(F)?
/)
(ANSWERS)(YES RIGHT)(NO WRONG)(END)

(QUESTION 4)
(CAPTION /4. WHAT ARE T(A) AND T(F) BOTH EQUAL TO?
(1)A      (2)F
(3)G      (4)S'      /)
(ANSWERS)(1 WRONG)(2 WRONG)(3 RIGHT)(4 WRONG)(END)

(QUESTION 5)
(CAPTION /5. WHAT IS T(G)?
(1) A       (2)F
(3)A AND F   (4)UNDEFINED
/)
(ANSWERS)(1 WRONG)(2 WRONG)(3 WRONG)(4 RIGHT)(END)
```

```
(QUESTION 6)
(CAPTION /6. X?S AND Y?S AND X=Y ? T(X)=  ?
(1) X                  (2)Y
(3)T(Y)                (4)UNDEFINED
/)
(ANSWERS)(1 WRONG)(2 WRONG)(3 RIGHT)(4 WRONG)(END)

(QUESTION 7)
(CAPTION /7. IF T IS ANY TRANSFORMATION FROM ANY SET S TO ANY
SET S', AND X?S AND Y?S, DOES X = Y ? T(X)=T(Y)?
/)
(ANSWERS)(YES RIGHT)(NO WRONG)(END)

(ACTION (1)(6 8)1)

(SECTION 2)
(DISPLAY 26)

(QUESTION 1)
(CAPTION /1. SUPPOSE WE DENOTE THIS TRANSFORMATION BY W.
WHAT IS W(C)?  (A,B,C,D,E,OR F)
/)
(ANSWERS)(A WRONG)(B WRONG)(C WRONG)(D WRONG)(E RIGHT)(END)

(QUESTION 2)
(CAPTION /2. WHAT IS W(D)?
(TYPE A,B,C,D,E, OR'U' FOR'UNDEFINED')
/)
(ANSWERS)(A WRONG)(B WRONG)(C WRONG)(D RIGHT)(U WRONG)(END)

(QUESTION 3)
(CAPTION /3. TO WHAT ELEMENT DOES THE ELEMENT TO WHICH
A CORRESPONDS CORRESPOND?   (A,B,C,D, OR E)
/)
(ANSWERS)(A WRONG)(B WRONG)(C WRONG)(D WRONG)(E RIGHT)(END)

(QUESTION 4)
(CAPTION /4. WHAT IS W(W(A))?   (B,C,D,E OR F)
/)
(ANSWERS)(B WRONG)(C WRONG)(D WRONG)(E RIGHT)(F WRONG)(END)

(QUESTION 5)
(CAPTION /5. IS THERE ANY DIFFERENCE BETWEEN W(W(A)) AND THE
ELEMENT WHICH IS THE CORRESPONDENT OF THE CORRESPONDENT
OF A?   /)
(ANSWERS)(YES WRONG)(NO RIGHT)(END)

(QUESTION 6)
(CAPTION /6. DO YOU AGREE NOW THAT GIVING TRANSFORMATIONS
NAMES AND DENOTING OBJECTS IN THE RANGE BY T(X) IS EASIER?
/)
(ANSWERS)(YES RIGHT)(NO WRONG (PLAY 99 1 1))(END)
```

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(QUESTION 7)
(CAPTION /7. WHAT IS W(W(D))?    (A,B,C,D,E,OR F)
/)
(ANSWERS)(A WRONG)(B WRONG)(C WRONG)(D RIGHT)(E WRONG)
         (F WRONG)(END)

(QUESTION 8)
(CAPTION /8. WHAT IS W(W(F))?  (A,B,C,D,E, OR F)
/)
(ANSWERS)(A WRONG)(B WRONG)(C WRONG)(D WRONG)
         (E WRONG)(F RIGHT)(END)

(QUESTION 9)
(CAPTION /9. DOES W(C) = W(E)?
/)
(ANSWERS)(YES RIGHT)(NO WRONG)(END)

(QUESTION 10)
(CAPTION /10. IS C = E?
/)
(ANSWERS)(YES WRONG)(NO RIGHT)(END)

(QUESTION 11)
(CAPTION /11. IF X AND Y ARE IN S, DOES W(X) = W(Y)
IMPLY X = Y?            /)
(ANSWERS)(YES WRONG)(NO RIGHT)(END)

(QUESTION 12)
(CAPTION /12. IF X AND Y ARE IN S, DOES X = Y
IMPLY W(X) = W(Y)?       /)
(ANSWERS)(YES RIGHT)(NO WRONG)(END)

(QUESTION 13)
(CAPTION /13. IF X AND Y ARE IN S AND X = Y,
WHY DOES W(X) = W(Y)? ----
(A) BECAUSE A = A AND B = B
(B) BECAUSE W IS A TRANSFORMATION
(C) BECAUSE THE DOMAIN OF W IS S.  (TYPE A,B,C)
/)
(ANSWERS)(A WRONG)(B RIGHT)(C WRONG)(END)

(QUESTION 14)
(CAPTION /14. WHAT IS W(W(W(A)))? (A,B,C,E OR F)
/)
(ANSWERS)(A WRONG)(B WRONG)(C WRONG)(E RIGHT)(F WRONG)(END)

(QUESTION 15)
(CAPTION /15. IS F = W(B)?        /)
(ANSWERS)(YES RIGHT)(NO WRONG)(END)
```

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(QUESTION 16)
(CAPTION /16. IS B = W(F)?      /)
(ANSWERS)(YES RIGHT)(NO WRONG)(END)

(QUESTION 17)
(CAPTION /17. COULD WE SAY THAT W "INTERCHANGES"
B AND F?      /)
(ANSWERS)(YES RIGHT)(NO WRONG(CAPTION/THINK ABOUT THIS/))(END)

(QUESTION 18)
(CAPTION /18. IS THERE ANY OTHER PAIR OF ELEMENTS OF S THAT W
"INTERCHANGES"?       /)
(ANSWERS)(YES WRONG)(NO RIGHT)(END)

(ACTION (2)(12 22)2)

(SECTION 3)
(DISPLAY 28)
(PLAY 100 1 8)(PLAY 101 1 8)

(QUESTION 1)
(CAPTION /1. WHAT IS T(C)? (A,E,F,G OR H)
(TO MAKE CAPITALS, PRESS THE BUTTON MARKED'UC'
BEFORE TYPING THE LETTER.)      /)
(ANSWERS)(A WRONG)(E WRONG)(F RIGHT)(G WRONG)(H WRONG)(END)

(QUESTION 2)
(CAPTION /2. WHAT IS THE IMAGE OF C UNDER T? (E,F,G OR H)
/)
(ANSWERS)(E WRONG)(F RIGHT)(G WRONG)(H WRONG)(END)

(QUESTION 3)
(CAPTION /3. DOES T(C) = IMAGE OF C UNDER T?     /)
(ANSWERS)(YES RIGHT)(NO WRONG)(END)

(QUESTION 4)
(CAPTION /4. WHAT IS THE IMAGE OF D UNDER T?
(E,F,G OR H)             /)
(ANSWERS)(E WRONG)(F WRONG)(G WRONG)(H RIGHT)(END)

(QUESTION 5)
(CAPTION /5. WHAT IS THE IMAGE OF F UNDER T?
(1)A        (2)B
(3)C        (4)D
(5)UNDEFINED           (TYPE NUMBER)
/)
(ANSWERS)(1 WRONG)(2 WRONG)(3 WRONG)(4 WRONG)
         (5 RIGHT)(END)
```

```
(QUESTION 6)
(CAPTION /6. WHAT SYMBOLIC SHORTHAND FORM WOULD WE USE FOR
"THE IMAGE OF A UNDER T IS E?"
(A) ATE            (B) T(A) = E
(C) T(E) = A       (D) A(T) = E
/)
(ANSWERS)(A WRONG)(B RIGHT)(C WRONG)(D WRONG)(END)

(QUESTION 7)
(CAPTION /7. WHAT SHORTHAND SYMBOLIC FORM WOULD WE USE FOR
"THE IMAGE OF B UNDER T IS G?"
(A) G = T(B)       (B) T(G) = B
(C) B(T) = G       (D) (T)B = G
/)
(ANSWERS)(A RIGHT)(B WRONG)(C WRONG)(D WRONG)(END)

(ACTION (3)(6 10)3)

(SECTION 4)
(DISPLAY 29)
(PLAY 102 1 8)

(QUESTION 1)
(CAPTION /1. WHAT IS U(2)?     (8,13 OR 37)
/)
(ANSWERS)(8 WRONG)(13 RIGHT)(37 WRONG)(END)

(QUESTION 2)
(CAPTION /2. WHAT IS U(5)?   (75,76,225,226)
/)
(ANSWERS)(75 WRONG)(76 RIGHT)(225 WRONG)(226 WRONG)(END)

(QUESTION 3)
(CAPTION /3. WHAT IS THE IMAGE OF 4 UNDER U?
(A) U(1)           (B) U(U(1))
(C) 25             (D) 145     (TYPE LETTER)
/)
(ANSWERS)(A WRONG)(B RIGHT)(C WRONG)(D WRONG)(END)

(QUESTION 4)
(CAPTION /4. WHAT IS U(U(1))?
(A) U(1)         (B) THE IMAGE OF 4 UNDER U
(C) 25           (D) 145
/)
(ANSWERS)(A WRONG)(B RIGHT)(C WRONG)(D WRONG)(END)

(QUESTION 5)
(CAPTION /5. WHAT IS THE IMAGE OF 6 UNDER U?
(A)109      (B)325      (C)UNDEFINED
/)
(ANSWERS)(A WRONG)(B WRONG)(C RIGHT)(END)
```

```
(QUESTION 6)
(CAPTION /6. HOW MANY ELEMENTS ARE THERE IN THE RANGE OF U?
(1,3,5,7,OR 79)     /)
(ANSWERS)(1 WRONG)(3 WRONG)(5 RIGHT)(7 WRONG)
          (79 WRONG)(END)

(QUESTION 7)
(CAPTION /7. HOW MANY ELEMENTS ARE THERE IN THE DOMAIN OF U?
(1,3,5,7,OR 79)      /)
(ANSWERS)(1  WRONG)(3 WRONG)(5 RIGHT)(7 WRONG)
          (79 WRONG)(END)

(QUESTION 8)
(CAPTION /8. HOW MANY ELEMENTS ARE THERE IN S'?
(1,3,5,7,OR 79)        /)
(ANSWERS)(1 WRONG)(3 WRONG)(5 WRONG)(7 WRONG)
          (79 RIGHT)(END)

(QUESTION 9)
(CAPTION /9. IS S' THE RANGE OF U?    /)
(ANSWERS)(YES WRONG)(NO RIGHT)(END)


(ACTION (4)(8 12)4)

(ACTION (1 2 3 4)(22 100) 1)

(CAPTION /THIS IS THE END OF THE FIRST LESSON!
/)


(FINIS)
```

APPENDIX III

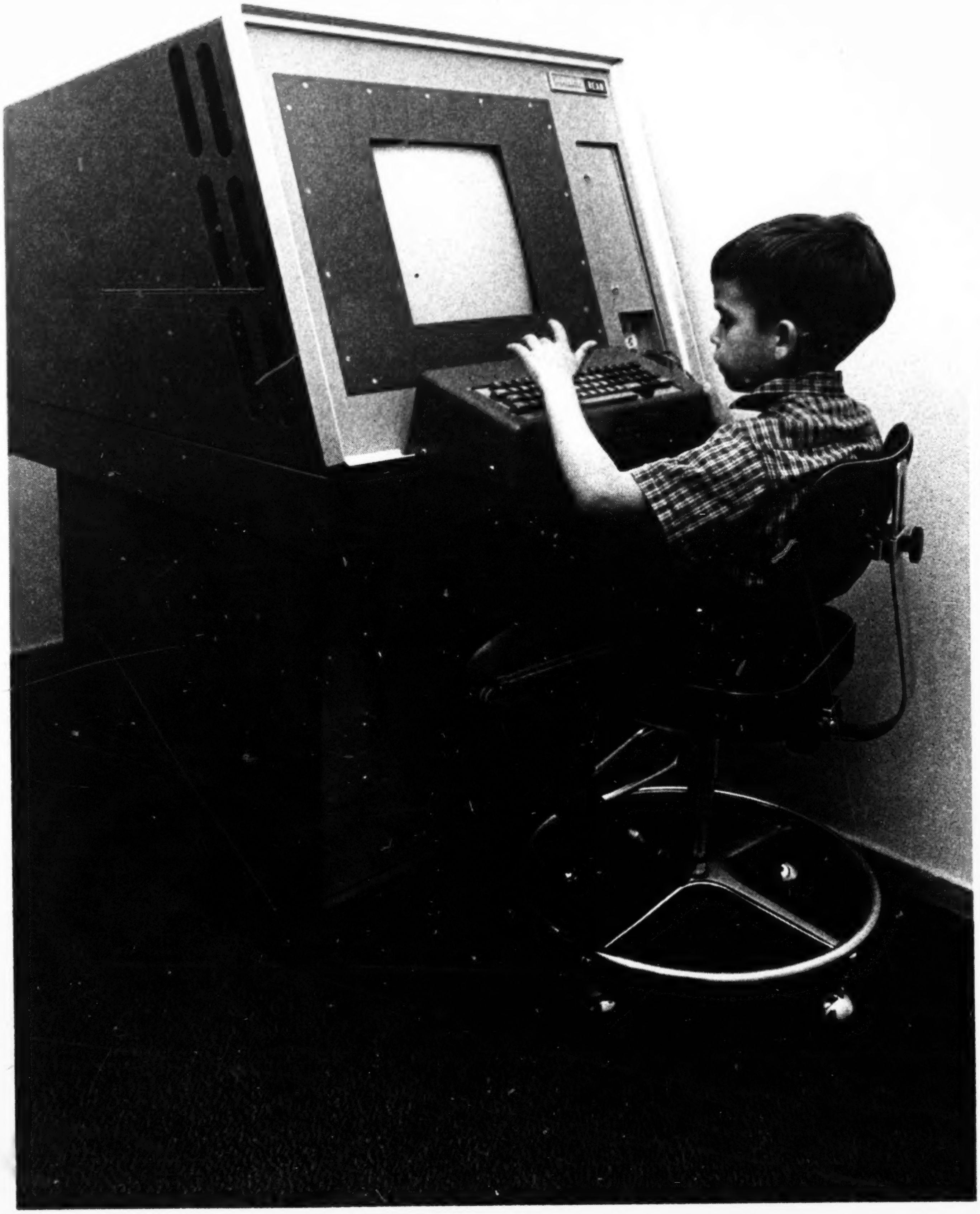
COMPUTER DISPLAYED GEOMETRIC DIAGRAMS AND TEXT

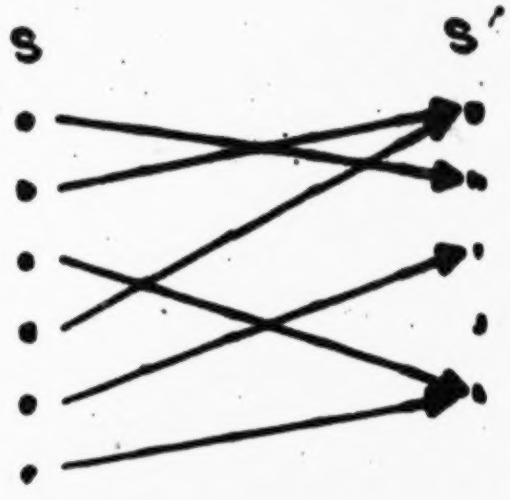
Selected Examples

Prepared by

Peter Belew

The following photographs taken from the scope illustrate almost all the diagrams in the first two lessons (see Appendix I).





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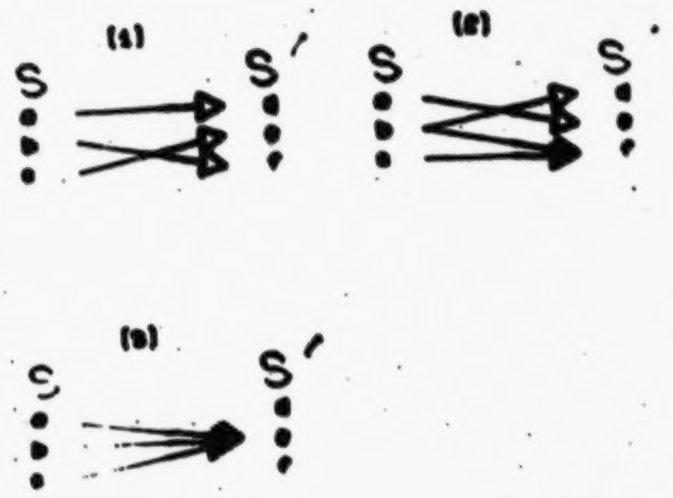

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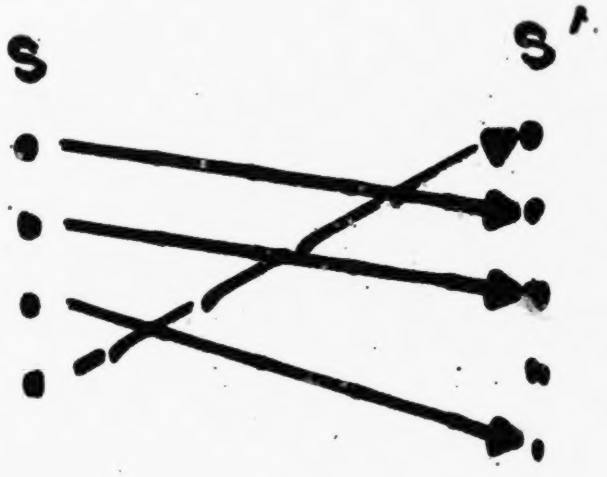

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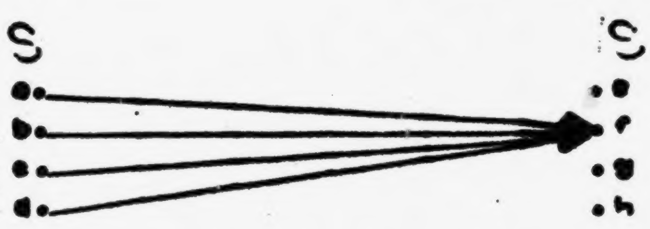

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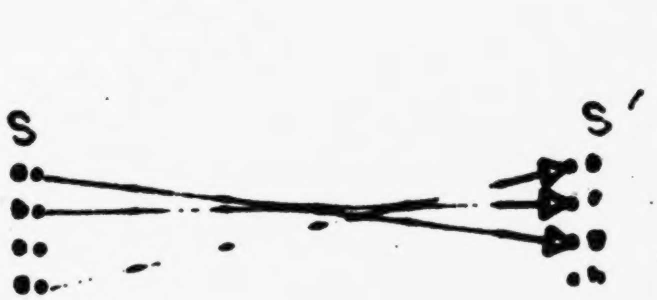

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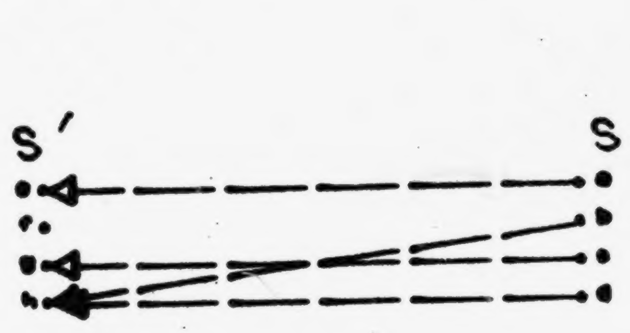

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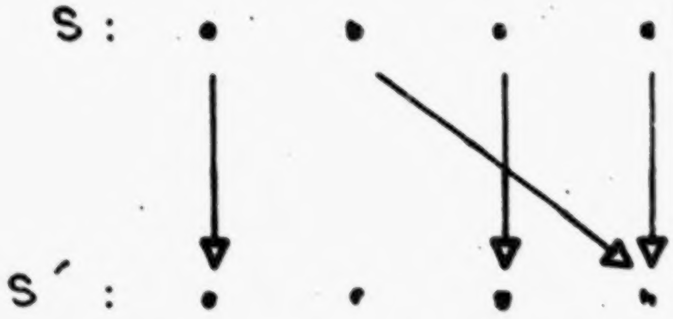

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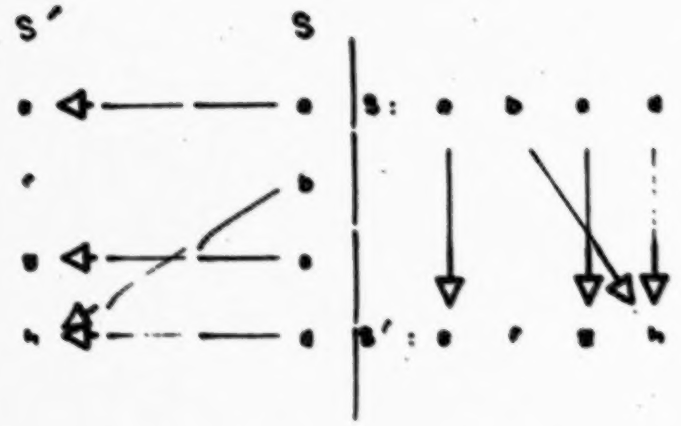

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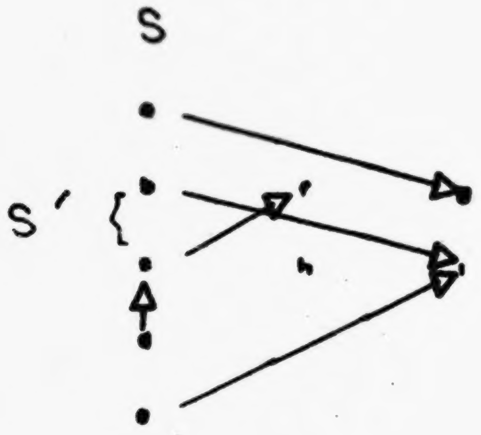

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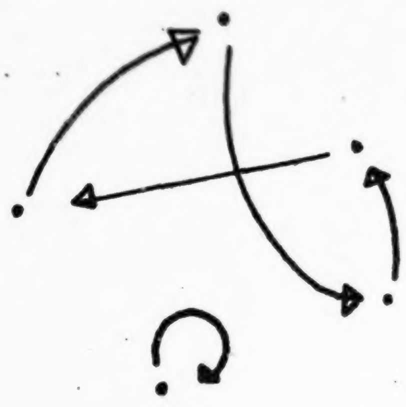

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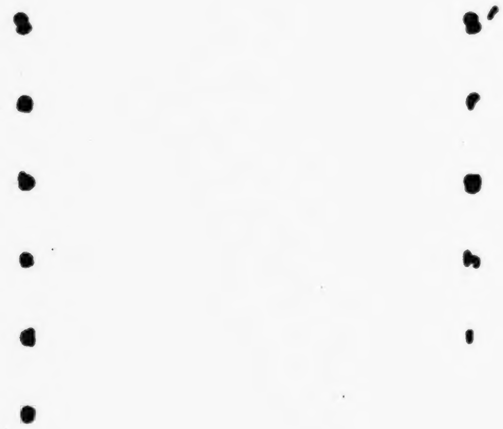

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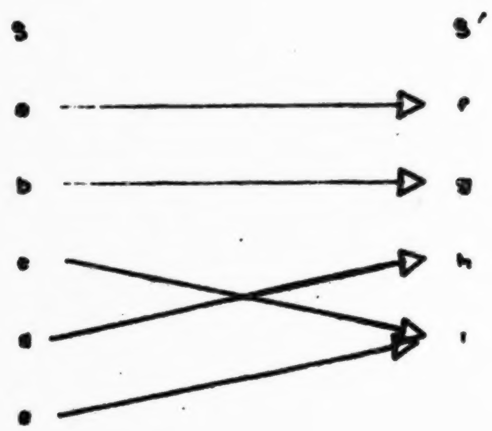

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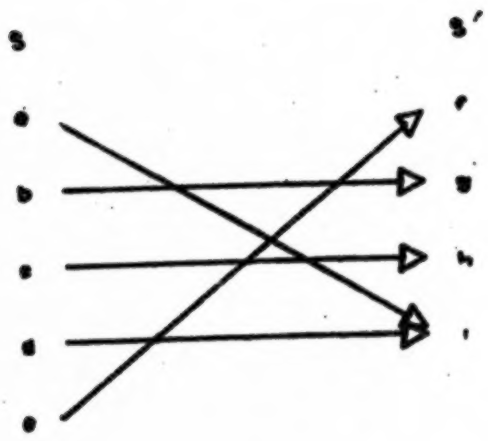
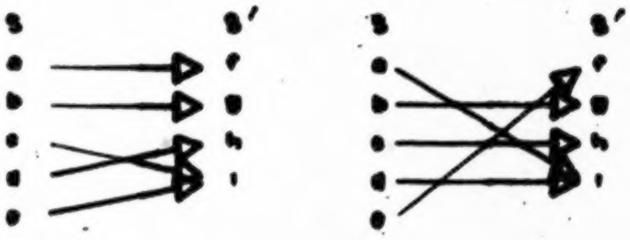

S
S′
a
b
c
d
e
f
g
h
i

page 78


S
S′
a
b
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f
g
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i
S
S′
a
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d
e
f
g
h
i

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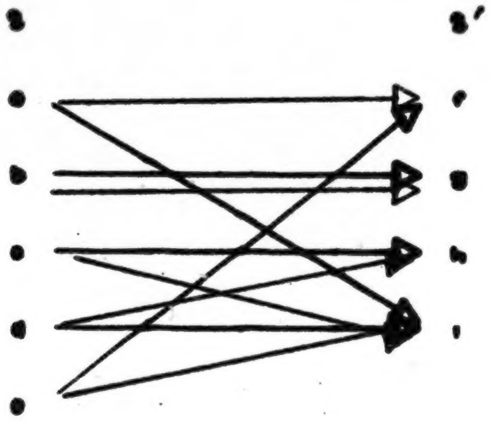
S
S′
a
b
c
d
e
f
g
h
i

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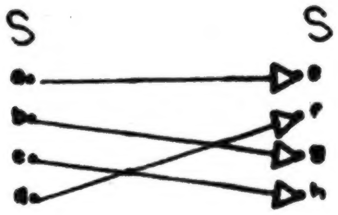
T(x) is read: 'T of x'

page 79


S
S
a
b
c
d
e
f
g
h
i

$T(a) = e$

$T(b) = g$

$T(c) = h$

$T(d) = f$

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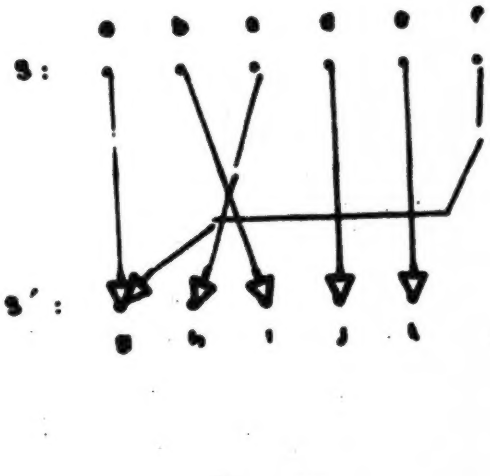
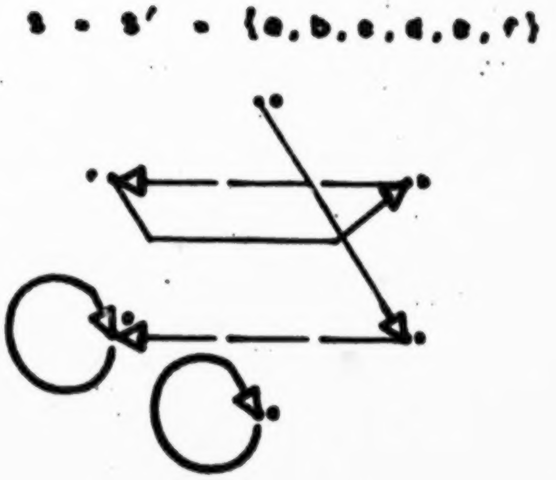

a
b
c
d
e
f
S:
S′:
g
h
i
j
k

page 80



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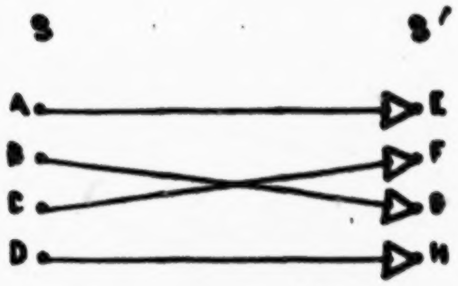
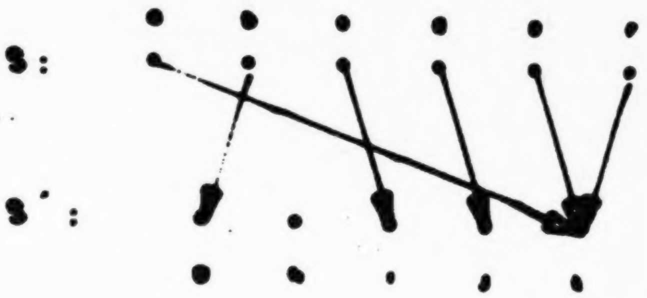

S
S′
A
B
C
D
E
F
G
H

page 82



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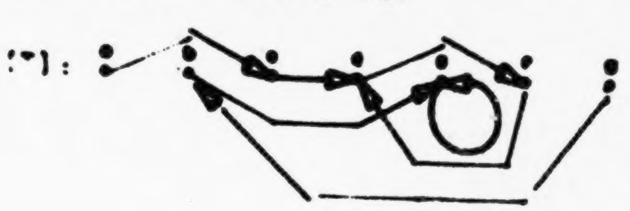
S=S′={a,b,c,d,e,f,g}
(T):

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$S=\{x \mid x$ was a president of the U.S. before L.B.J.$\}$

$S'=\{0,1,2,3,4,5\}$

$f(x)=$ number of times x was elected.

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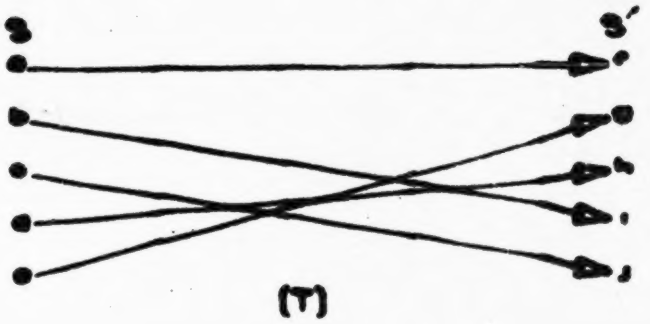

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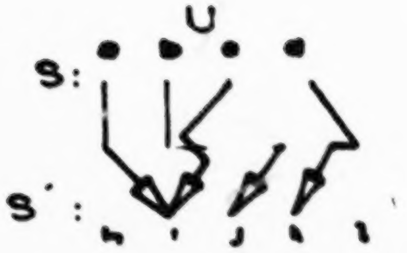

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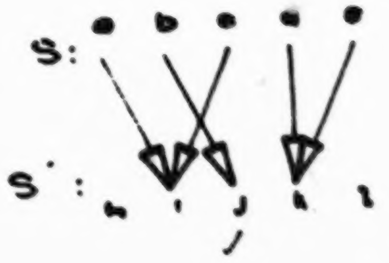

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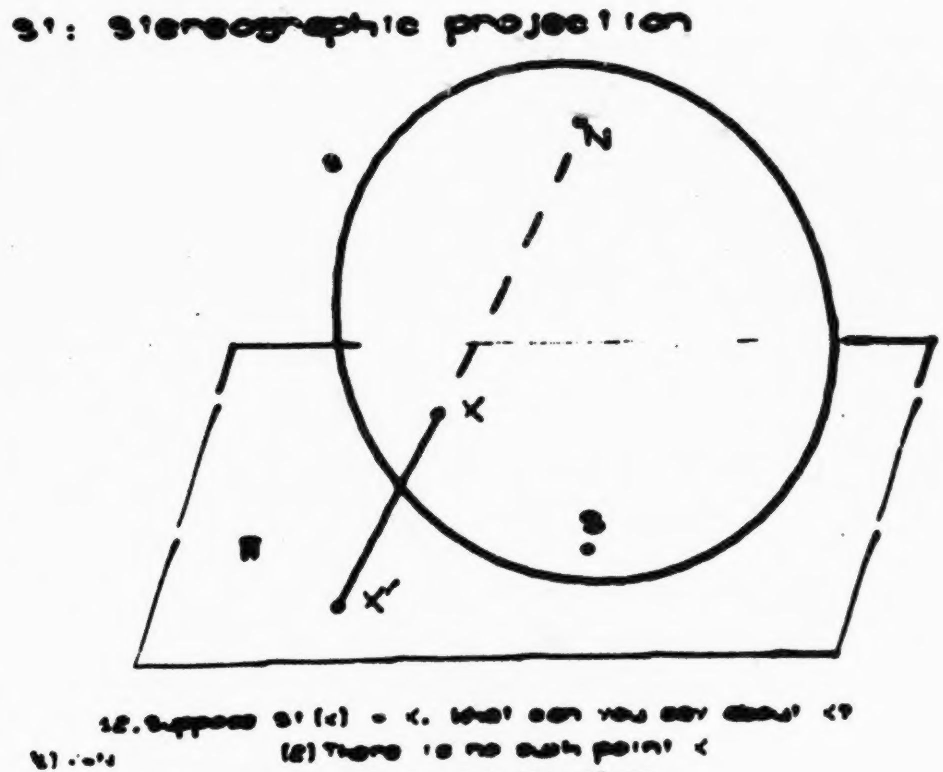

S': Stereographic projection

12. Suppose S'(x) = x. What can you say about x?

(1) x=N (2) There is no such point x

(3) x is on both the sphere and the plane

ANSWER:

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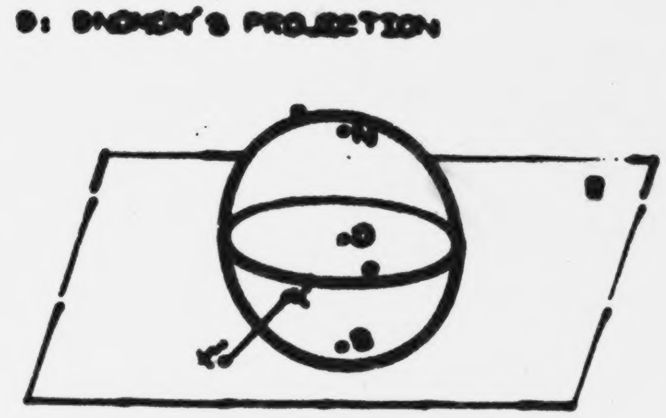
D: GNOMON'S PROJECTION
N
O
S
X'
X''

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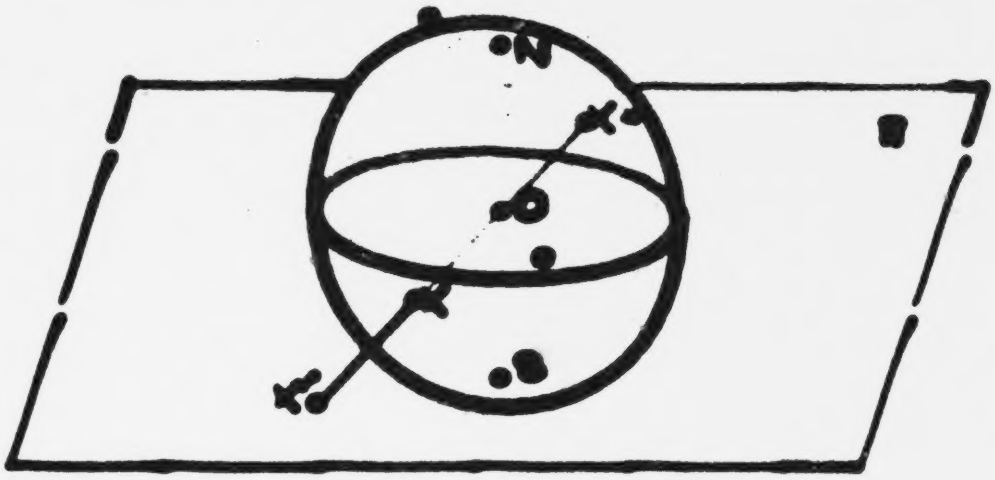

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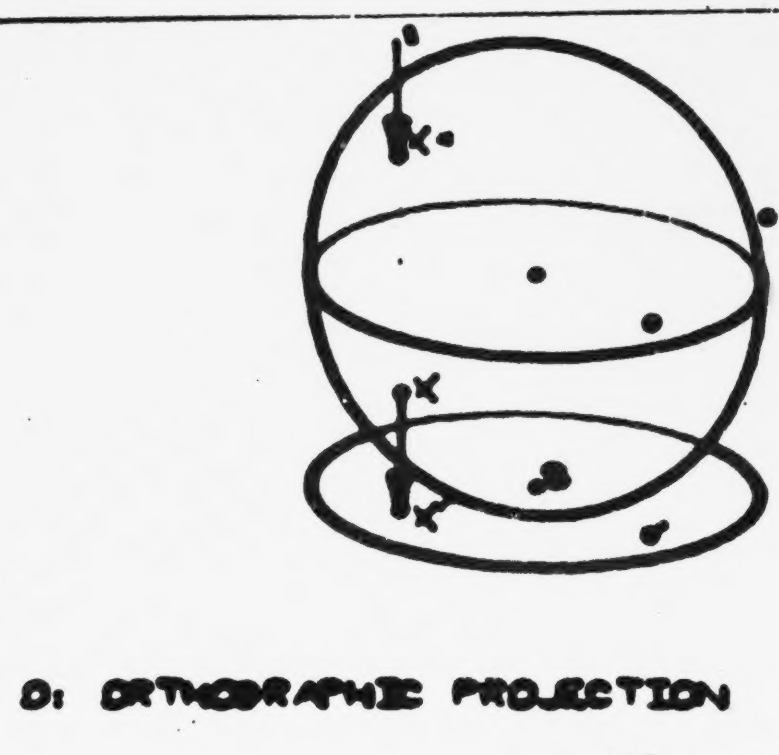
X'
X
X''
S
D: ORTHOGRAPHIC PROJECTION

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APPENDIX IV

TSA TEXT OF THE
MULTIPLE CHOICE PROGRAM

THE INTERPRETER USED
FOR THE LESSON OF APPENDIX II

Prepared by

Peter Belew

```
BEGIN COMMENT "THIS IS THE LESSON INTERPRETER";
DATAFILE(426,"LESONS");
DPYFILE(203,"PICTUR");COMMENT"REC 65 HAS TEXT CONTROL WORDS";
ZILCH←CONVERT("//");
PICEND←0;  COMMENT"END OF PICTURE PORTION OF BUFFER";
ARRAY BOOL[3];
 COMMENT"BOOL AND VAL ARE USED FOR CONDITIONAL
        STATEMENTS";
PROCEDURE DISPFUN(X);
BEGIN   Y←X[1];
        IF Y="COMMENT" THEN RETURN
        ELSE IF Y="PLAY"THEN BEGIN PLAY(X[2],X[3],X[4]);
                 RETURN END
        ELSE IF Y="WAIT"THEN BEGIN WAIT(X[2]);RETURN END
        ELSE IF Y="CLEAR" THEN BEGIN DPYEND←0;RETURN END
        ELSE IF Y="SET" THEN BEGIN BOOL[X[2]]←TRUE;RETURN END
        ELSE IF Y="UNSET"THEN BEGIN BOOL[X[2]]←FALSE;
            RETURN END
        ELSE IF Y="DISPLAY"THEN BEGIN
            DPYEND←0;
            PICTURE(X[2]);
            PICEND←DPYEND END
        ELSE IF Y="ADDPIC" THEN BEGIN
            PICTURE(X[2]);
            PICEND←DPYEND END
        ELSE IF Y="CAPTION" THEN BEGIN
            LA←LENGTH(X);
            FOR LB←2 STEP 1 UNTIL LA DO BEGIN
            LOC←X[LB];
            IF LOC="SUB" THEN PICTURE(72)
            ELSE IF LOC="SUPER"THEN PICTURE(73)
            ELSE IF LOC="NOTEPS" THEN PICTURE(71)
            ELSE IF LOC="PARALLEL" THEN PICTURE(74)
            ELSE IF LOC="PERP" THEN PICTURE(75)
            ELSE CAPTION(LOC)
            END END
        ELSE IF Y="RESTORE" THEN
            BEGIN DPYEND←PICEND;WAIT(1)END
        ELSE BEGIN CAPTION(Y);
        CAPTION("...WE CAN'T DO THAT!");
        END;
        DISPLAY(DPYEND);
        RETURN
END;

COMMENT"THE FOLLOWING IS A PROCEDURE FOR ESTABLISHING EQUALITY
OF LISTS AS SETS--IT IS NOT RECURSIVE";
PROCEDURE EQL(A,B);
```

```
COMMENT "HERE THE PROGRAM BEGINS";
START:   CR(15);
VAL← FALSE; FOR I←1STEP 1 UNTIL 3 DO-BOOL[I]←FALSE;I←NIL;
TYPE("  WHEN AN ANSWER IS REQUESTED,
TYPING '←' SKIPS THE QUESTION,
AND TYPING 'X' ALLOWS YOU TO CORRECT YOUR ANSWER
");
TYPE("
WHAT PAGE DO YOU WANT TO START AT?
  IN LESSON 1:(1,8,15 OR 22)
  IN LESSON 2: 30,37, OR 51");CR(1);
TIMER←60;RR;P←CONVERT(ANSWER);
P←IF KIND(P)=1 THEN P ELSE 1;TYPE("...ZOT!");
DATA (1,P,1);P←NIL;
NEWTOPIC:ARRAY LOCSEC[15],LOC[2];
 TOPPAGE←PAGE(1);TOPLINE←LINE(1);
LA3COMP:THING←*1; DING←THING[1];
         IF DING="STOP" THEN GO TO EXIT;
         IF DING="TOPIC" THEN
         BEGIN
              TOPNO←THING[2];
              TOPPAGE←PAGE(1);
              TOPLINE←LINE(1);
              GO TO LABCOMP
         END;
         IF DING ="LOCSEC"THEN
              BEGIN LOCSEC←THING[2];GO TO LOCFOUND END;
         IF DING ="SECTION"THEN
         BEGIN
              IDX←THING[2];
              LOC[1]←PAGE(1);
              LOC[2]←LINE(1);
              SIZ←LENGTH(LOCSEC);
         IF SIZ<IDX THEN BEGIN
              TEMP←LOCSEC;
              ARRAY LOCSEC[IDX];
              FOR I←1 STEP 1 UNTIL SIZ DO LOCSEC[I]←TEMP[I];
              END;
         LOCSEC[IDX]←LOC
         END;
IF THING[1]?"FINIS" THEN GO TO LABCOMP;
LOCFOUND:     THING←NIL;IDX←NIL;LOC←NIL;TEMP←NIL;
SIZ←LENGTH(LOCSEC);
ARRAY RIGHT[SIZ],ERRORS[SIZ],CURRENTQUES[2];
```

```
 DATA(1,TOPPAGE,TOPLINE); AGAIN←FALSE;VAL←TRUE;

 PROG:THING←*1; FUNCT←THING[1];
  ADVPROG:IF FUNCT="QUESTION"THEN BEGIN
          CURRENTQUES[1]←PAGE(1);
          CURRENTQUES[2]←LINE(1);
          QUESNO←THING[2];
          AGAIN ←FALSE; REPEAT←FALSE;TRIALS←0;VAL←TRUE;
          MAX←LENGTH(THING);
          MAXWRONG←2;
          TIMRR←60;
          FOR I←3 STEP 1 UNTIL MAX-1 DO
          BEGIN IF THING[I]="TIME"THEN
               TIMRR←THING[I+1]
               ELSE IF THING[I]="TRIALS"THEN
               MAXWRONG←THING[I+1]
          END
          END
 ELSE IF FUNCT="GOTO" THEN BEGIN
          LAB←THING[2];
          DATA (1,LOCSEC[LAB][1],LOCSEC[LAB][2])
          END
 ELSE IF FUNCT="IF" THEN BEGIN
          DING←THING[2];      PLACE←THING[5];
          BUL←IF KIND(DING)=1 THEN BOOL[DING]
           ELSE IF DING="RIGHT"THEN VAL
           ELSE IF DING="WRONG" THEN ?VAL   ELSE FALSE;
          IF BUL
          THEN DATA(1,LOCSEC[PLACE][1],LOCSEC[PLACE][2])
          END
 ELSE IF FUNCT="ACTION"THEN BEGIN
          LABELLIST←THING[2] ; SCORE←0;MAX←LENGTH(LABELLIST);
          MAXX←LENGTH(ERRORS);
          FOR I←1 STEP 1 UNTIL MAX DO
               BEGIN IND←LABELLIST[I];
               IF IND?MAXX THEN
               BEGIN INCR←ERRORS[IND];
               IF KIND(INCR)=1 THEN
               SCORE←SCORE+INCR
               END END;
          IND←NIL; INCR←NIL;I←NIL;MAX←NIL;MAXX←NIL;
          IF SCORE>THING[3][2] THEN BEGIN
               CAPTION("TOO MANY ERRORS
CALL TEACHER
");
               GO TO TEACH END;
          IF SCORE?THING[3][1] THEN BEGIN
               COMMENT"GO TO LABELLED SECTION";
               LAB←THING[4];
               DATA(1,LOCSEC[LAB][1],LOCSEC[LAB][2]);
               GO TO PROG END
```

```
        END
ELSE IF FUNCT="SECTION" THEN BEGIN
        SECNO←THING[2]  ;
        RIGHT[SECNO]←0;ERRORS[SECNO]←0 END
ELSE IF FUNCT="ANSWERS" THEN BEGIN
        X←(LENGTH(THING)=2);
        QUESEND←DPYEND;
        CAPTION (IF AGAIN THEN " TRY AGAIN!"
            ELSE " ANSWER!");
  OOPS:  DING←IF X THEN "(" ELSE ZILCH;  P←IF X THEN 1 ELSE 0;
        SLASHES←0;
  KRUG:  LENGTH←1;TIMER←TIMRR;RR;
            CAPTION(ANSWER);
 IF ANSWER="←" THEN GO TO ADVANCE;
 IF ANSWER="X" THEN GO TO OOPS;
            TIMRR←TIMRR-LATENCY;
            IF KIND(ANSWER)=2?ANSWER?ZILCH THEN
            IF X THEN DING←
                IF ANSWER?" "?ANSWER?"," THEN
                DING&ANSWER&" "ELSE DING
            ELSE DING←IF ANSWER?" "?DING?ZILCH
                THEN DING&ANSWER ELSE DING;
            IF ANSWER="/"THEN SLASHES←
                IF SLASHES=0 THEN 1 ELSE 0
            ELSE IF SLASHES=0 THEN
            IF ANSWER="("THEN P←P+1
                ELSE IF ANSWER=")"THEN P←
                IF P?0 THEN -100ELSE
                   P-1;
            IF ANSWER?ZILCH??TIMEOUT THEN
                GO TO KRUG;
        IF X THEN BEGIN DING←DING & ")"; P←P-1 END;
        IF P=0?SLASHES=0?DING?ZILCH THEN
            BEGIN Y←CONVERT(DING);
            IF KIND(Y)?2 THEN DING←Y END;
DPYEND←QUESEND;CAPTION("ANSWER = ",DING);
        P←NIL; SLASHES←NIL;
        IF TIMEOUT THEN BEGIN
            IF REPEAT THEN BEGIN
            CAPTION("TIMEOUT,CALL TEACHER
");
            WAIT(1);DPYEND←QUESEND;
            ERRORS[SECNO]←ERRORS[SECNO]+1;
            GO TO TEACH END;
            AGAIN←TRUE;
            REPEAT←TRUE;
            CAPTION("TIMEOUT!");
            WAIT(1);DPYEND←QUESEND;
            ERRORS[SECNO]←ERRORS[SECNO]+1;
            DATA(1,CURRENTQUES[1],CURRENTQUES[2]+1);
            END
```

```
         ELSE BEGIN
         FOUND←FALSE;SIZ←FALSE;
         ANSLUP:    THING←*1;OTVJET←THING[1];
              IF OTVJET="END" THEN
                    IF FOUND THEN GO TO ONWARD
                    ELSE GO TO NOTFOUND;
              IF EQL(?OTVJET,?DING) THEN
                    BEGIN FOUND←TRUE;ITEM←THING END;
              IF OTVJET="ELSE" THEN BEGIN
                    SIZ←TRUE;TEMP←THING END;
              GO TO ANSLUP;
COMMENT"AT THIS POINT THE REPLY DOES NOT
         MATCH ANY ANSWER ON THE LIST";
NOTFOUND:     IF SIZ THEN BEGIN
              ITEM←TEMP; TEMP←NIL; GO TO ONWARD END;
         CAPTION("
THIS ANSWER WAS NOT ON THE LIST.");
         WAIT(1);DPYEND←QUESEND;
         DATA (1,CURRENTQUES[1],CURRENTQUES[2]+1);
         AGAIN←TRUE;
         GO TO PROG;
         ONWARD:DING←NIL;VALIDITY←ITEM[2];SIZ←LENGTH(ITEM);
         IF VALIDITY="RIGHT"THEN BEGIN
              AGAIN←FALSE;
              CAPTION("RIGHT!
");
              FOR I←3 STEP 1 UNTIL SIZ DO DISPFUN(?ITEM[I]);
              WAIT(1);DPYEND←PICEND;WAIT(2);
               COMMENT"CLEAR FOR NEXT QUESTION";
              RIGHT[SECNO]←RIGHT[SECNO]+2 END
         ELSE IF VALIDITY="WRONG"THEN BEGIN
              VAL←FALSE;
              AGAIN←TRUE;
              CAPTION("WRONG!
");
              FOR I←3 STEP 1 UNTIL SIZ DO DISPFUN(?ITEM[I]);
              WAIT(1);DPYEND←QUESEND;
              IF TRIALS=0 THEN ERRORS[SECNO]←ERRORS[SECNO]+2;
              TRIALS←TRIALS+1;
              IF TRIALS?MAXWRONG THEN BEGIN
                    CAPTION("TOO MANY TRIALS
CALL TEACHER
");
                    GO TO TEACH END;
              DATA (1,CURRENTQUES[1],CURRENTQUES[2]+1)
         END
         ELSE IF REPEAT THEN
              BEGIN CAPTION
                   ("ALMOST RIGHT,
CALL TEACHER
");
```

```
                   FOR I←3 STEP 1 UNTIL SIZ DO
                       DISPFUN(?ITEM[I]);
                   WAIT(1);DPYEND←QUESEND;
                   ERRORS[SECNO]←ERRORS[SECNO]+1;
                   GO TO TEACH END
             ELSE BEGIN REPEAT←TRUE;AGAIN←TRUE;
                   VAL←FALSE;
                   ERRORS[SECNO]←ERRORS[SECNO]+1;
                   CAPTION("ALMOST RIGHT.
");

                   FOR I←3STEP 1 UNTIL SIZ DO
                       DISPFUN(?ITEM[I]);
                   WAIT(1); DPYEND←QUESEND;
                   DATA(1,CURRENTQUES[1],CURRENTQUES[2]+1);
             END
        END
END
ELSE IF FUNCT="FINIS" THEN GO TO NEWTOPIC
ELSE IF FUNCT="STOP"  THEN GO TO EXIT
ELSE IF FUNCT="TOPIC" THEN GO TO PROG
ELSE IF FUNCT="LOCSEC" THEN GO TO PROG
ELSE IF FUNCT="COMMENT" THEN GO TO PROG
ELSE IF FUNCT="PLAY" ? ?AGAIN THEN DISPFUN(?THING);
GO TO PROG;

 ADVANCE:THING←*1; FUNCT←THING[1];
 IF FUNCT="QUESTION"?FUNCT="SECTION"?FUNCT="TOPIC"?
  FUNCT="CAPTION"? FUNCT="ADDPIC"?FUNCT="RESTORE"
  ?FUNCT="PLAY"?FUNCT="CLEAR"?FUNCT="DISPLAY"?FUNCT="IF"?
  FUNCT="GOTO"?FUNCT="ACTION"?FUNCT="FINIS"?FUNCT="STOP"
 THEN BEGIN DPYEND←PICEND; GO TO ADVPROG END
 ELSE GO TO ADVANCE;

TEACH:   DPYEND←PICEND;
AGAIN←FALSE;
CAPTION("
I AM THE TEACHER!
SECTION",SECNO,"  QUESTION",QUESNO);
EXIT:    J←LENGTH(RIGHT);   PLUS←0;     MINUS←0;
FOR I←1 STEP 1 UNTIL J DO BEGIN
        IF RIGHT[I]?NIL THEN PLUS←PLUS+RIGHT[I];
        IF ERRORS[I]?NIL THEN MINUS←MINUS+ERRORS[I] END;
CAPTION("
TOTAL RIGHT SCORE IS",PLUS,"
TOTAL WRONG SCORE IS",MINUS,"
");
CAPTION ("IN 5 SECONDS, WE START AGAIN");
WAIT(5);GO TO START
END.
```

APPENDIX D.--ERIC REPORT RESUME

OE 6000 (REV. 9-66)

DEPARTMENT OF HEALTH EDUCATION AND WELFARE
OFFICE OF EDUCATION

ERIC REPORT RESUME

(TOP)

ERIC ACCESSION NO.

CLEARINGHOUSE ACCESSION NUMBER

RESUME DATE 6-1-68

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TITLE

DEVELOPING NEW MATERIALS FOR HIGH SCHOOL GEOMETRY FINAL REPORT

PERSONAL AUTHOR(S)

Dana S. Scott

INSTITUTION (SOURCE)

Stanford University, Stanford, California, Institute for Mathematical Studies in the Social Sciences

SOURCE CODE

REPORT/SERIES NO.

OTHER SOURCE

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CONTRACT GRANT NUMBER OE 6-10-021

PAGINATION ETC.

153 pages

RETRIEVAL TERMS

High School Geometry
Geometric Transformations
Computer-Assisted Instruction

IDENTIFIERS

PDP-1 Computer, Philco CRT Consoles, TSA Prog. System

ABSTRACT

Geometric transformations have for some time been regarded as an area suitable for enrichment and development of geometry both for high school and teacher training curricula. This project studied the material from the point of view of the high school level. The plan was to develop a high school course both for use in the classroom and for use with a computer controlled system of programmed lessons. Many difficulties were encountered in trying to organize and prepare the material for the machine. One of the main problems had to do with the display and control of figures. The conclusion reached was that the solution to the problem is possible, but a reasonably extensive series of lessons would require a larger computer than we had available. Furthermore, the heavy task of program preparation required a larger staff than we could support on this project. A certain amount of technical development in graphical display still remains to be done for the kind of problem also. In the meantime, work needs to be done in developing the proper materials. For the area of geometric transformations we propose in our final report an outline for a one semester high school course. The short term of the project did not allow us to make arrangements for teaching the material to high school students, something that surely should be done in the near future.